On the duality of proofs and countermodels in labelled sequent calculi

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From mathematical practice to formal logic

Proving a theorem vs. finding a counterexample

In logic, analytic calculi may reduce the proving of theorems to *automatic* tasks.

Completeness theorems guarantee a perfect *duality* between proofs in a formal systems and well defined types of counterexamples (countermodels).

However, completeness proofs are often non-effective (non-constructive) and countermodels are artificial (Henkin sets or Lindenbaum algebra) and far from what we regard as counterexamples.

Furthermore, canonical countermodels provided by traditional completeness proofs may fall out from the intended classes and need a model theoretic fine tuning.

Can we find "concrete" countermodels in the same automated way in which we find proofs?

Refutation calculi (Dyckhoff and Pinto, Skura, Goranko, Fiorentini et al., Gore', ...) build refutations rather than proofs and can be used as a basis for building countermodels. These calculi are separate from the direct inferential systems, rules are not invertible (root-first the rules give sufficient conditions of non-validity), sometimes use a pre-processing of formulas in a suitable normal form.

Tableaux (Kripke, Fitting, ...) restrict the refutations to relational models and countermodels can be read off from failed proof search. Expressive power limited to relatively few logics and non-locality of the rules make the extraction of the countermodel a non immediate task.

Unifying proof search and countermodel construction

- The method is a synthesis of:
  1. Generation of calculi with internalized semantics
  2. A Tait-Schütte-Takeuti style completeness proof
  3. A syntactic counterpart of semantic filtration *or* a suitable proof-theoretic embedding.
- We present a countermodel-generating calculus for intuitionistic logic, the method of finitization through truncation and through a faithful embedding
- We indicate how it is extended to other non-classical logics (e.g. intermediate, multi-modal, provability) and beyond geometric theories.
- We conclude with some open problems and further directions.

Semantics in logical calculi

- **Implicit**: *Sequent calculus for classical logic*, *display calculi* (Wansing), *nested sequents* (Kashima 1994), *tree-sequents* (Cerrato 1996), *deep sequents* (Brünnler 2006, Stouppa 2007), *tree-hypersequents* (Poggiolesi 2008), *hypersequents, non-deterministic matrices* (Avron, Konikowska, Zamansky, Ciabattoni, et al.).
- **Explicit**: *Labelled sequents* (Mints 1997, Viganó 2000, Kushida and Okada 2003, Castellini and Smaill 2002, Castellini 2005), *labelled tableaux* (Fitting 1983, Catach 1991, Nerode 1991, Goré 1998, Massacci 2000, Orlowska and Golińska Pilarek 2011), *labelled natural deduction* (Fitch 1966, Simpson 1994, Basin, Matthews, Viganó 1998), *hybrid logic* (Blackburn 2000, Bolander, Braüner), *Labelled Deductive Systems* (Gabbay, Russo, et al. 1996).

1. Formal Kripke semantics in contraction-free sequent calculi

The starting building block is the sequent calculus **G3c**.

- Introduced by Ketonen and successively improved and extended by Kleene, Dragalin, Troelstra (cf. *Basic Proof Theory*)
- The rules are invertible
- Not only cut but also weakening and contraction are admissible
- Shared context rules
- Suited for root-first proof search
- Multisuccedent sequents allow uniform treatment of classical and intuitionistic logic

The calculus **G3c**

Initial sequents:

$P, \Gamma \Rightarrow \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \& B, \Gamma \Rightarrow \Delta}\ L\&$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \& B}\ R\&$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Structural properties of **G3c**

- *All the rules of* **G3c** *are invertible, with height-preserving inversion.* E.g.: If $\vdash_n \Gamma \Rightarrow \Delta, A\&B$, then $\vdash_n \Gamma \Rightarrow \Delta, A$ and $\vdash_n \Gamma \Rightarrow \Delta, B$.
- *The structural rules of weakening and contraction are height-preserving admissible in* **G3c**:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\, LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\, RC$$

- *Cut is admissible in* **G3c**.

Root-first determinism, no need of backtracking in proof search.

Rule systems with labels

Proof systems exploiting the characterization of a logic in terms of Kripke semantics appear in several guises, with the following in common:

- Explanation of modal operators through semantically justified introduction and elimination rules.
- Properties of Kripke frames through rules for the accessibility relation.

We internalize the accessibility relation of Kripke frames in a G3-style sequent calculus to obtain cut- and contraction free systems in a uniform way (Negri 2005)

- Add possible worlds as labels for formulas $x : A$
- Obtain the rules for the logical constants by unfolding the inductive definition of truth at a world
- Add properties of the accessibility relation xRy as *rules*, following the method of “axioms as rules” (Negri and von Plato 1998)

Intuitionistic propositional logic

$$x \Vdash A \supset B \iff \textit{for all } \ y,\ x \leqslant y \textit{ and } \ y \Vdash A \textit{ implies } \ y \Vdash B$$

- $$\frac{x \leqslant y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \supset B} R\supset$$

  variable condition: y not (free) in Γ, Δ

- $$\frac{x : A \supset B, x \leqslant y, \Gamma \Rightarrow \Delta, y : A \quad y : B, x : A \supset B, x \leqslant y, \Gamma \Rightarrow \Delta}{x : A \supset B, x \leqslant y, \Gamma \Rightarrow \Delta} L\supset$$

The system **G3I**

Initial sequents: $x \leqslant y, x : P, \Gamma \Rightarrow \Delta, y : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \& B, \Gamma \Rightarrow \Delta}\, L\&$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \& B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{x \leqslant y, x : A \supset B, \Gamma \Rightarrow y : A, \Delta, \quad x \leqslant y, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset$$

$$\frac{x \leqslant y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Order rules:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref$$

$$\frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\, Trans$$

Observe that **G3I** does not have the restriction of a single-succedent premiss in the $R\supset$ rule.

The calculus does not become classical:

$$\dfrac{\dfrac{x \leqslant y, y : P \overset{\color{red}?}{\Rightarrow} x : P, y : \bot}{\Rightarrow x : P, x : \neg P}\, R\supset}{\Rightarrow x : P \vee \neg P}\, R\vee$$

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The calculus does not become classical:

$$\dfrac{\dfrac{\overset{?}{x \leqslant y, y : P \Rightarrow x : P, y :\bot}}{\Rightarrow x : P, x : \neg P} \; R\supset, \; y \textit{ fresh}}{\Rightarrow x : P \vee \neg P} \; R\vee$$

All the rules are invertible and thus “preserve countermodels”; terminal node in a failed proof search gives a Kripke countermodel:

$$\begin{array}{l} \bullet \; y \Vdash P \\ \uparrow \\ \bullet \; x \nVdash P \end{array}$$

The parallel proof search/countermodel construction works in full generality for the **G3K**-based modal labelled calculi (Negri 2009). Let us see in detail how, in the case of intuitionistic logic.

2. A Tait-Schütte-Takeuti style completeness proof

Consider a derivation in **G3I**:

Let K be a frame with accessibility a reflexive and transitive accessibility relation $\mathcal{R}$.

W the set of world labels used in derivations in **G3I**.

Interpretation of W in $K \equiv [\![\cdot]\!] : W \to K$.

Valuation of atomic formulas $\mathcal{V} : \mathit{AtFrm} \to \mathcal{P}(K)$

$k \in \mathcal{V}(P)$ iff $k \Vdash P$.

Valuations for intuitionistic Kripke semantics are requested to satisfy the *monotonicity property*

$$k\mathcal{R}k' \text{ and } k \Vdash P \text{ implies } k' \Vdash P.$$

They are extended to arbitrary formulas by the following inductive clauses:

- $k \Vdash \bot$ for no k;
- $k \Vdash A\&B$ if $k \Vdash A$ and $k \Vdash B$;
- $k \Vdash A \vee B$ if $k \Vdash A$ or $k \Vdash B$;
- $k \Vdash A \supset B$ if for all k', from $k\mathcal{R}k'$ and $k' \Vdash A$ follows $k' \Vdash B$.

$\Gamma \Rightarrow \Delta$ *true* for a given interpretation of labels and valuation of propositional variables in a frame, if for all labelled formulas $x : A$ and relational atoms yRz in Γ, if $[\![x]\!] \Vdash A$ and $[\![y]\!]\mathcal{R}[\![z]\!]$ in K, then for some $w : B$ in Δ, $[\![w]\!] \Vdash B$.

A sequent is *valid* if it it true for every interpretation and every valuation of propositional variables in the frame.

Validity: If sequent $\Gamma \Rightarrow \Delta$ is derivable in **G3I**, then it is valid in every reflexive and transitive frame.

Proof: Initial sequents are valid and rules preserve validity.

Completeness: Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of **G3I**. Then either the sequent is derivable in **G3I** or it has a Kripke countermodel.

Proof: We define for an arbitrary sequent $\Gamma \Rightarrow \Delta$ in the language of **G3I** a reduction tree by applying root first the rules of **G3I** in all possible ways. If the construction terminates we obtain a proof, else we obtain an infinite tree. By *König's lemma* an infinite tree has an infinite branch, which is used to define a countermodel to the endsequent.

Construction of the countermodel:
Let $\Gamma_0 \Rightarrow \Delta_0 \equiv \Gamma \Rightarrow \Delta, \Gamma_1 \Rightarrow \Delta_1 \ldots, \Gamma_i \Rightarrow \Delta_i, \ldots$ be the infinite branch.
Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i$$

$$\mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$, and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$.

Frame $K \equiv$ labels appearing in the relational atoms in $\mathbf{\Gamma}$

Relation $R \equiv$ all the xRy's in $\mathbf{\Gamma}$.

Construction of the reduction tree imposes the frame properties of the countermodel: Because of the rules *Ref* and *Trans*, the constructed frame is reflexive and transitive.

Valuation: For all the atomic formulas P such that $x \leqslant y$, $x : P$ in $\mathbf{\Gamma}$, set $y \Vdash P$, and for all atomic formulas $z : Q$ in $\mathbf{\Delta}$, set $z \nVdash Q$.

Finally show inductively on the weight of formulas:

- If $x : A$ is in $\mathbf{\Gamma}$, then A is forced in the model at node x
- If $x : A$ is in $\mathbf{\Delta}$, then A is not forced in the model at node x.

Therefore we have a countermodel to the endsequent $\Gamma \Rightarrow \Delta$.

The proof search defined in the proof of the completeness theorem may produce infinite branches and therefore infinite countermodels.

As an example, consider the search for a proof of the law of double negation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A, w : \neg A}}{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A} L\supset}{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A} R\supset}{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A} L\supset}{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A, y : \neg A} R\supset}{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A} L\supset}{x \leqslant y, y : \neg\neg A \Rightarrow y : A} Ref}{\Rightarrow x : \neg\neg A \supset A} R\supset
$$

There are two ways to prove that our looping staircase cannot produce a derivation, a *minimality argument* or the construction of a *finite countermodel*, a suitable truncation of the infinite countermodel provided by the general completeness proof.

Minimality argument: If the sequent were derivable, suppose that the topmost in the attempted proof search has a derivation of height n. By the hp-substitution $[z/w]$ we obtain a derivation of the same height of the sequent

$$y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant z, y : \neg\neg A, z : A, z : A \Rightarrow y : A, z : \neg A$$

and thus, by hp-contraction, of

$$y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A.$$

A step of *Ref* gives a derivation of height $n + 1$ of

$$y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A.$$

But this sequent had derivation height $n + 2$ so the derivation has been shortened by one step. It is therefore useless to proceed with steps that lead to duplications of formulas modulo re-labelling.

$$
\begin{array}{c}
\vdots\ \mathcal{D}[z/w],\ \textit{hp-contr} \\
{\color{red}n} \quad y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A \\
\hline
{\color{red}n+1} \quad y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A \quad \textit{Ref} \\
\hline
{\color{red}n+2} \quad y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A \quad L{\supset} \\
\hline
{\color{red}n+3} \quad y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A, y : \neg A \quad R{\supset} \\
\hline
{\color{red}n+4} \quad y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A \quad L{\supset} \\
\hline
{\color{red}n+5} \quad x \leqslant y, y : \neg\neg A \Rightarrow y : A \quad \textit{Ref} \\
\hline
{\color{red}n+6} \quad \Rightarrow x : \neg\neg A \supset A \quad R{\supset}
\end{array}
$$

shortened by one step!

3. A syntactic counterpart of semantic filtration

Finite countermodel construction:

Block the procedure of proof-search until a suitable *saturation* condition is met.

For a sequent $\Gamma \Rightarrow \Delta$ in the proof search, let $\downarrow \Gamma$ ($\downarrow \Delta$) be the union of the antecedents (succedents) in the branch below $\Gamma \Rightarrow \Delta$.

For any label x and sequent $\Gamma \Rightarrow \Delta$ we define two increasing sets of formulas $\mathcal{F}_\Gamma(x)$ and $\mathcal{F}_\Delta(x)$ that correspond to the formulas labelled by x in the sequent.

How to obtain a saturated sequent:
Apply all the possible rules except in case $z : A \supset B$ is a labelled formula in Δ and for some earlier label y (i.e. some label y such that $y \leqslant z$ is in $\downarrow \Gamma$) we have $\mathcal{F}_{\downarrow\Gamma}(z) \subseteq \mathcal{F}_{\downarrow\Gamma}(y)$ and $\mathcal{F}_{\downarrow\Delta}(z) \subseteq \mathcal{F}_{\downarrow\Delta}(y)$.
When such situation occurs we call $z : A \supset B$ a *looping formula*.

Which is the looping formula in the previous proof search?

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A, w : \neg A}\ L\supset}{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A}\ R\supset}{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A}\ L\supset}{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A}\ R\supset}{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A, y : \neg A}\ L\supset}{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A}\ Ref}{x \leqslant y, y : \neg\neg A \Rightarrow y : A}\ R\supset}{\Rightarrow x : \neg\neg A \supset A}
$$

$$
\begin{array}{c}
\vdots \\
\underline{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A, w : \neg A} \; L\supset \\
\underline{y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \Rightarrow y : A} \; R\supset \\
\underline{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A, z : \neg A} \; L\supset \\
\underline{y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \Rightarrow y : A} \; R\supset \\
\underline{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A, y : \neg A} \; L\supset \\
\underline{y \leqslant y, x \leqslant y, y : \neg\neg A \Rightarrow y : A} \; Ref \\
\underline{x \leqslant y, y : \neg\neg A \Rightarrow y : A} \; R\supset \\
\Rightarrow x : \neg\neg A \supset A
\end{array}
$$

Here $w : \neg A$ is the looping formula and the red formulas show the relevant inclusions between the sets of labelled formulas.

Finite countermodel construction:
Proceed as in the completeness proof but instead of building the countermodel on the infinite branch, we build it on the saturated sequent $\Gamma \Rightarrow \Delta$ and take the two finite sets $\downarrow \Gamma, \downarrow \Delta$ in place of the (potentially) infinite $\mathbf{\Gamma}$, $\mathbf{\Delta}$.

We then show inductively on the weight of formulas that A is forced in the model at node z if $z : A$ is in $\downarrow \Gamma$ and A is not forced at node z if $z : A$ is in $\downarrow \Delta$.

Therefore we have a countermodel to the endsequent.

Decidability through completeness: Proof search may produce infinitely many labels, however, there is only a finite number of available formulas, so eventually, after a finite (with predictable upper bound) number of steps, either a derivation or a saturated sequent (hence a countermodel) is obtained.

Extensions

The completeness proof can be extended to systems with *geometric* and *generalized geometric* frame conditions (Negri 2009, 2013).

These cover

- **intermediate logics** (Dyckhoff and Negri 2012)
- **most classical modal logics**
- all the **displayable non-classical logics** (by Kracht's characterization in terms of *primitive* frame conditions, which are geometric)
- **multi-modal systems**, such as the *logic of distributed knowledge* and the *logic of group belief* (Hakli and Negri 2008, 2012)
- **multi-dimensional modal systems**, such as *knowability logic* (Maffezioli, Naibo and Negri 2012). Beyond geometric frame conditions.

Intuitionistic multi-modal logics

- The procedure we have seen does not work if we have, say, *seriality* instead of reflexivity. This is the case for *deontic logic* or for *temporal logic*. Instead of truncating the countermodel (which would have a terminal node and thus would not be serial) we add a loop in correspondence of the looping label, i.e. add a relation $x \leqslant y$ for $\mathcal{F}(x) \subseteq \mathcal{F}(y)$.
- More generally, it is possible to consider systems of (intuitionistic) multimodal logic and use the procedure of finitization of the countermodel construction to solve problems of decidability in modal logic (Garg, Genovese and Negri, 2012).

G3IM: A sequent calculus for intuitionistic (multi-)modal logic

Extends **G3I** by the rules:

$$\frac{y : A, x : \Box_a A, xR_a y, \Gamma \Rightarrow \Delta}{x : \Box_a A, xR_a y, \Gamma \Rightarrow \Delta} \, L\Box_a \qquad \frac{xR_a y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_a A} \, R_a\Box_a$$

$$\frac{xR_a z, x \leqslant y, yR_a z, \Gamma \Rightarrow \Delta}{x \leqslant y, yR_a z, \Gamma \Rightarrow \Delta} \, Mon_{R_a}$$

$$\frac{xR_a x', \{x_i R_a x_i'\}_i, \Gamma \Rightarrow \Delta}{\{x_i R_a x_i'\}_i, \Gamma \Rightarrow \Delta}$$

Properties of **G3IM**

- All the structural properties of G3-sequent calculi
- Soundness w.r.t. Kripke semantics
- Weak subformula property

however, proof search may not terminate ...

$$
\begin{array}{cl}
\vdots & \\
x_0 : \Box(\Box A \supset A), x_2 : \Box A \supset A \Rightarrow x_2 : \Box A & \\
\hline
x_0 : \Box(\Box A \supset A), x_2 : \Box A \supset A \Rightarrow x_2 : A & L\supset \\
\hline
x_0 R x_2 ; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_2 : A & L\Box \\
\hline
x_0 R x_1 , x_1 R x_2 ; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_2 : A & \mathit{Trans} \\
\hline
x_0 R x_1 ; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_1 : \Box A & R\Box \\
\hline
x_0 R x_1 ; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_1 : A & L\supset \\
\hline
x_0 R x_1 ; x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_1 : A & L\Box \\
\hline
x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_0 : \Box A & R\Box \\
\hline
x_0 \leq x_0 ; x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_0 : A & L\supset \\
\hline
x_0 R x_0 , x_0 \leq x_0 ; x_0 : \Box(\Box A \supset A) \Rightarrow x_0 : A & L\Box
\end{array}
$$

Properties of **G3IM**

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however, proof search may not terminate ...

$$
\begin{array}{cl}
\vdots & \\
x_0 : \Box(\Box A \supset A), x_2 : \Box A \supset A \Rightarrow x_2 : \Box A & \\
\hline
x_0 : \Box(\Box A \supset A), x_2 : \Box A \supset A \Rightarrow x_2 : A & L\supset \\
\hline
x_0 R x_2; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_2 : A & L\Box \\
\hline
x_0 R x_1, x_1 R x_2; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_2 : A & \mathit{Trans} \\
\hline
x_0 R x_1; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_1 : \Box A & R\Box \\
\hline
x_0 R x_1; x_0 : \Box(\Box A \supset A), x_1 : \Box A \supset A \Rightarrow x_1 : A & L\supset \\
\hline
x_0 R x_1; x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_1 : A & L\Box \\
\hline
x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_0 : \Box A & R\Box \\
\hline
x_0 \leq x_0; x_0 : \Box(\Box A \supset A), x_0 : \Box A \supset A \Rightarrow x_0 : A & L\supset \\
\hline
x_0 R x_0, x_0 \leq x_0; x_0 : \Box(\Box A \supset A) \Rightarrow x_0 : A & L\Box
\end{array}
$$

Key insights

All worlds in a sequent $\Gamma \Rightarrow \Delta$ obtained during backward proof search lie on a rooted, directed tree, whose edges are relations introduced by the rules R$\supset$ and R$\Box$(the *label-producing* rules).
$\ll$ is the “earlier in the tree” relation.

$\mathcal{F}_S(x)$ $\sim$ formulas labelled by x in S.

Saturation modulo looping: For every label-producing rule on x, either the branch is closed under the rule, or there is some early label $y \ll x$ with $\mathcal{F}_S(x) \subseteq \mathcal{F}_S(y)$ ($x \preccurlyeq y$). Applies also to seriality.

This forces termination: If n is the size of the endsequent, there are at most 2^n distinct elements in a chain $x_0 \ll x_1 \ll \ldots$.

Countermodel obtained by closing under the additional preorder $x \preccurlyeq y$.

Scope of the method

A representative collection of logics shown constructively decidable by the method; also seriality can be added:

Logic	Axiom	Frame conditions
K	–	–
T	$\Box_a A \supset A$	$\forall x.\ xR_a x$
K4	$\Box_a A \supset \Box_a \Box_a A$	$\forall xyz\ (xR_a y \& yR_a z \supset xR_a z)$
S4	Axioms of K4 and T	Conditions of K4 and T
I	$\Box_a A \supset \Box_b \Box_a A$	$\forall xyz\ (xR_b y \& yR_a z \supset xR_a z)$
unit	$A \supset \Box_a A$	$\forall xy\ (xR_a y \supset x \leq y)$
subsumption	$\Box_b A \supset \Box_a A$	$\forall xy\ (xR_a y \supset xR_b y)$

Internalizing finiteness: provability logics

- If R is a transitive *irreflexive* Noetherian (equiv. every chain is finite) relation, then, for all x,

$$x \Vdash \Box A \iff \textit{for all } y, xRy \textit{ and } y \Vdash \Box A \textit{ implies } y \Vdash A$$

- If R is a transitive *reflexive* Noetherian (every chain eventually becomes stationary) relation, then, for all x,

$$x \Vdash \Box A \iff \textit{for all } y, xRy \textit{ and } y \Vdash \Box(A \supset \Box A) \textit{ implies } y \Vdash A$$

The forcing condition for the modality in Noetherian frames justifies the rules:

- To obtain a sequent calculus for **GL** replace the $R\Box$ rule of **G3K** with

$$\frac{xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box\text{-}GL$$

- To obtain a sequent calculus for **Grz** replace it with

$$\frac{xRy, y : G(A), \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box\text{-}Grz$$

  where $G(A) \equiv \Box(A \supset \Box A)$

then **G3GL**(**G3Grz**) is simply like **G3K** with $R\Box$ replaced by $R\Box$-GL($R\Box$-Grz) and *Ref* and *Trans* added.

How provability logic enforces termination

Consider the following failed derivation in **G3K4**:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{xRy, yRz, xRz, z : \neg(\Box P\&\neg P), x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}}{xRy, yRz, xRz, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}\, L\Box}{xRy, yRz, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}\, \mathit{Trans}}{xRy, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, y : \Box P}\, R\Box \quad \triangledown}{xRy, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, y : \Box P\&\neg P}\, R\&}{xRy, y : \neg(\Box P\&\neg P), x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P}\, L\neg}{xRy, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P}\, L\Box}{x : \Box\neg(\Box P\&\neg P) \Rightarrow x : \Box P}\, R\Box
$$

In **G3GL** termination is enforced: **no loops on boxed formulas.**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{xRy, yRz, xRz, y : \Box P, z : \Box P, z : \neg(\Box P\&\neg P), x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}}{xRy, yRz, xRz, y : \Box P, z : \Box P, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}\, L\Box}{xRy, yRz, y : \Box P, z : \Box P, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, z : P}\, \mathit{Trans}}{xRy, y : \Box P, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, y : \Box P}\, R\Box\text{-}GL \quad \triangledown}{xRy, y : \Box P, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P, y : \Box P\&\neg P}\, R\&}{xRy, y : \Box P, y : \neg(\Box P\&\neg P), x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P}\, L\neg}{xRy, y : \Box P, x : \Box\neg(\Box P\&\neg P) \Rightarrow y : P}\, L\Box}{x : \Box\neg(\Box P\&\neg P) \Rightarrow x : \Box P}\, R\Box\text{-}GL
$$

Proof-search in **G3GL** gives a finite decision procedure without loop-checking.

For **G3Grz** an almost local blocking condition suffices to end the proof search and find finite countermodels

There is a semantic embedding of **Int** into **GL**. No known syntactic proof of faithfulness.

Syntactic embedding of **Int** into **Grz**

Using a variant of Gödel's translation

$$
\begin{array}{rcl}
P^{\Box} & := & \Box P \\
\bot^{\Box} & := & \bot \\
(A \supset B)^{\Box} & := & \Box(A^{\Box} \supset B^{\Box}) \\
(A \& B)^{\Box} & := & A^{\Box} \& B^{\Box} \\
(A \vee B)^{\Box} & := & A^{\Box} \vee B^{\Box}
\end{array}
$$

G3I $\vdash \Gamma \Rightarrow \Delta$ if and only if **G3Grz** $\vdash \Gamma^{\Box} \Rightarrow \Delta^{\Box}$.
((Dyckhoff and Negri 2013).

The embedding of **G3I** into **G3Grz** gives a finite decision procedure and finite countermodel generation for **Int**.
Blocking condition for $R\Box$-Grz: If the succedent contains both $x : \Box B$ and $x : B$, and the antecedent contains, for some x_0, both $x_0 R x$ and $x_0 : G(B)$, we do not further analyze $x : \Box B$; similarly if $x : B$ is in any sequent in the branch.

The elimination of logical constants in the axiom-to-rules conversion

Logical constant	Left rule	Right rule
$\&$	,	branching
$\vee$	branching	,
$\supset$	split in succ./ant.	split in ant./succ.
positive $\exists$ (geometric axiom)	variable condition	—
negative $\forall$ (cogeometric axiom)	—	variable condition
noetherianity	modified $L\Box$ rule	modified $R\Box$ rule
quantifier alternations beyond $\forall\exists$	?	?

A motivating problem: Knowability logic

Epistemic conceptions of truth justify the *knowability principle*:

If A is true, then it is possible to know that A $\quad A \supset \Diamond \mathcal{K} A$ (**KP**)

The Church-Fitch paradox: formal derivation that poses minimal assumptions on the alethic and epistemic operators, and that starts from the knowability principle to conclude *(collective) omniscience*:

All truths are actually known $\quad A \supset \mathcal{K} A$ (**OP**)

The paradox was presented by Fitch (1963) but found by Joe Salerno and Julien Murzi to have actually been suggested by Church in a series of referee's reports that date back to 1945 (Salerno 2009).

The paradox has given rise to a flourishing and ever expanding literature (can be found even in social networks). The main goal has been to show that the paradox does not affect an *intuitionistic* conception of truth.

The derivation of the paradox is indeed done in classical logic. Intuitionistic logic proves its negative version, but to prove intuitionistic *underivability* of the positive version, a careful proof analysis is needed.

Knowability logic

Intuitionistic logic blocks the derivation of **OP** from the Moore-instance ($A \& \neg \mathcal{K} A$) of **KP**. Is this enough? No! *Derivability* vs. *admissibility*.

So the goal has been to develop a proof theory for *knowability logic*: a cut-free sequent system for bimodal logic extended by the knowability principle.

The knowability principle does not reduce to atomic instances, so it cannot be translated into rules through the methodology of “axioms as rules”.

A labelled calculus for knowability logic

Extend **G3I** by rules for $\mathcal{K}$ and $\Diamond$:

- $x \Vdash \mathcal{K}A$ iff for all y, $xR_{\mathcal{K}}y$ implies $y \Vdash A$
- $x \Vdash \Diamond A$ iff for some y, $xR_{\Diamond}y$ and $y \Vdash A$

The clauses are converted into rules:

$$\frac{y : A, x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta}{x : \mathcal{K}A, xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta}\ L\mathcal{K} \qquad \frac{xR_{\mathcal{K}}y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{K}A}\ R\mathcal{K}$$

$$\frac{xR_{\Diamond}y, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\ L\Diamond \qquad \frac{xR_{\Diamond}y, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma \Rightarrow \Delta, x : \Diamond A}\ R\Diamond$$

In $R\mathcal{K}$ and $L\Diamond$, y does not appear in Γ and Δ

From (modal) axioms to (frame) rules

Recall that various extensions are obtained by adding the frame properties that correspond to the added axioms, for example

Logic	Axiom	Frame property	Rule
T	$\Box A \supset A$	$\forall x\, xRx$ reflexivity	$\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}$
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}$
D	$\Box A \supset \Diamond A$	$\forall x \exists y\, xRy$ seriality	$\dfrac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, y$
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	$\dfrac{yRw, zRw, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\, w$
W	$\Box(\Box A \supset A) \supset \Box A$	trans., irref., and no infinite R-chains	modified $\Box$ rules

but *knowability* is different from all such cases...

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\overline{\Rightarrow x : A \supset \Diamond \mathcal{K} A}$$

Finding the right rules for knowability logic

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$$\dfrac{\overline{x \leqslant y, y : A \Rightarrow y : \Diamond \mathcal{K} A}}{\Rightarrow x : A \supset \Diamond \mathcal{K} A}\ R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\cfrac{\cfrac{\overline{x \leqslant y, yR_{\Diamond}z, y : A \Rightarrow y : \Diamond\mathcal{K}A}}{x \leqslant y, y : A \Rightarrow y : \Diamond\mathcal{K}A}\; Ser_{\Diamond}}{\Rightarrow x : A \supset \Diamond\mathcal{K}A}\; R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$\dfrac{\dfrac{\dfrac{\overline{x \leqslant y, yR_{\diamond}z, y : A \Rightarrow y : \diamond\mathcal{K}A, z : KA}}{x \leqslant y, yR_{\diamond}z, y : A \Rightarrow y : \diamond\mathcal{K}A}\ R\diamond}{x \leqslant y, y : A \Rightarrow y : \diamond\mathcal{K}A}\ Ser_{\diamond}}{\Rightarrow x : A \supset \diamond\mathcal{K}A}\ R\supset$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame condition and the rules needed, by root-first proof search:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{x \leqslant y, yR_\diamond z, zR_\mathcal{K} w, y : A \Rightarrow y : \diamond\mathcal{K}A, w : A}}{x \leqslant y, yR_\diamond z, y : A \Rightarrow y : \diamond\mathcal{K}A, z : KA}\,R\mathcal{K}}{x \leqslant y, yR_\diamond z, y : A \Rightarrow y : \diamond\mathcal{K}A}\,R\diamond}{x \leqslant y, y : A \Rightarrow y : \diamond\mathcal{K}A}\,Ser_\diamond}{\Rightarrow x : A \supset \diamond\mathcal{K}A}\,R\supset
$$

Finding the right rules for knowability logic

The calculus itself is used to find the frame conditions and the rules needed, by root-first proof search:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant y, \overline{y \leqslant w}, yR_{\diamond}z, zR_{\mathcal{K}}w, \overline{y : A} \Rightarrow y : \diamond\mathcal{K}A, \overline{w : A}}{x \leqslant y, yR_{\diamond}z, zR_{\mathcal{K}}w, y : A \Rightarrow y : \diamond\mathcal{K}A, w : A}\,\diamond\mathcal{K}\text{-}Tr}{x \leqslant y, yR_{\diamond}z, y : A \Rightarrow y : \diamond\mathcal{K}A, z : KA}\,R\mathcal{K}}{x \leqslant y, yR_{\diamond}z, y : A \Rightarrow y : \diamond\mathcal{K}A}\,R\diamond}{x \leqslant y, y : A \Rightarrow y : \diamond\mathcal{K}A}\,Ser_{\diamond}}{\Rightarrow x : A \supset \diamond\mathcal{K}A}\,R\supset$$

the uppermost sequent is derivable by *monotonicity*.

Finding the right rules for knowability logic (cont.)

The two extra-logical rules used are:

$$\frac{xR_\Diamond y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ser}_\Diamond \qquad \frac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}{xR_\Diamond y, yR_\mathcal{K} z, \Gamma \Rightarrow \Delta}\ \Diamond\mathcal{K}\text{-}\mathit{Tr}$$

$\mathit{Ser}_\Diamond$ has the condition $y \notin \Gamma, \Delta$. The rules correspond to the frame properties

$$\forall x \exists y. xR_\Diamond y \qquad \mathbf{Ser}_\Diamond$$

$$\forall x \forall y \forall z (xR_\Diamond y \,\&\, yR_\mathcal{K} z \supset x \leqslant z) \qquad \Diamond\mathcal{K}\text{-}\mathbf{Tr}$$

The universal frame property $\Diamond\mathcal{K}$**-Tr** is, however, **too strong**: The instance of rule $\Diamond\mathcal{K}$-*Tr* used in the derivation of **KP** is not applied (root first) to an arbitrary sequent, but to one in which the middle term is the eigenvariable introduced by $\mathit{Ser}_\Diamond$.

So we have the requirements:

- *$\Diamond\mathcal{K}$-Tr has to be applied above $\mathit{Ser}_\Diamond$*
- The middle term of $\Diamond\mathcal{K}$-*Tr* is the eigenvariable of $\mathit{Ser}_\Diamond$.

Finding the right rules for knowability logic (cont.)

The move to consider the two rules not independently of each other, but as a *system of rules*, coupled together by the side condition on the eigenvariable.

With this proviso, the system of rules is equivalent to the frame property

$$\forall x \exists y (x R_{\Diamond} y \,\&\, \forall z (y R_{\mathcal{K}} z \supset x \leqslant z)) \quad \textbf{KP-Fr}$$

KP-Fr is derivable in a **G3**-sequent system for intuitionistic first-order logic extended by the two rules $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-*Tr* with the side condition.

Conversely, any derivation that uses rules $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-*Tr* *in compliance with the side condition*, can be transformed into a derivation that uses cuts with **KP-Fr**.

The system with rules $\Diamond\mathcal{K}$-*Tr* and $Ser_\Diamond$ that respect the side condition is a cut-free equivalent of the system that employs ***KP-Fr*** as an axiomatic sequent in addition to the structural rules.

The rules that correspond to **KP-Fr do not** follow the geometric rule scheme. However, all the structural rules are still admissible in the presence of such rules. In particular, cut elimination holds and the proof follows the usual pattern;
a genuine extension of the method of conversion of axioms into rules.

The system obtained by the addition of suitable combinations of these two rules provides a complete contraction- and cut-free system for the *knowability logic* **G3KP**, that is, intuitionistic bimodal logic extended with **KP**.

Intuitionistic solution to Fitch's paradox through an exhaustive proof analysis in **KP**: **OP** is **not** derivable in **G3KP**.
End of motivating problem. More about the Church-Fitch paradox in Maffezioli, Naibo, and Negri (2012).

Generalizing geometric implications

Recall that a *geometric implication* (or geometric axiom) is a formula of the form

$$\forall \overline{x}(A \supset B)$$

where A and B do not contain $\supset$ or $\forall$

Normal form for geometric implications:

$$\forall \overline{x}(\& P_i \supset \exists y_1 M_1 \vee \cdots \vee \exists y_n M_n) \qquad GA$$

P_i atomic formulas, M_j conjunctions of atomic formulas $Q_{j_1} \& \ldots \& Q_{j_{k_j}}$, y_j not free in the P_i.

Geometric implications are taken as the base case in the inductive definition of generalized geometric implications.

$GA_0 \equiv GA \qquad GRS_0 \equiv GRS$

$GA_1 \equiv \forall \overline{x}(\& P_i \supset \exists y_1 \& GA_0 \vee \cdots \vee \exists y_m \& GA_0)$

$GA_{n+1} \equiv \forall \overline{x}(\& P_i \supset \exists y_1 \& GA_n \vee \cdots \vee \exists y_m \& GA_n)$

here $\& GA_i$ denotes a conjunction of GA_i-axioms.

Systems of rules for generalized geometric implications

In Negri (2003) we have established an equivalence between the axiomatic systems based of geometric axioms and contraction- and cut-free sequent systems with geometric rules.

The equivalence can be extended by a suitable definition of *systems of rules* for generalized geometric axioms.

Word “system” used in the same sense as in linear algebra: systems of equations with **variables in common**, and each equation is meaningful and can be solved only if considered together with the other equations of the system. Here in addition the rules are subject to the condition of appearing in a given **order** in a derivation.

Example of GA_1 axiom and GRS_1 system of rules

Axiom of join semi-lattices:

$$\forall xy \exists z((x \leqslant z \,\&\, y \leqslant z) \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)) \qquad \textit{lub-A}$$

Observe that the axiom can be equivalently written in the form

$$\forall xy \exists z \forall w((x \leqslant z \,\&\, y \leqslant z) \,\&\, (x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))$$

This as an axiom of the form GA_1, with the first antecedent of atomic formulas empty. As system of rules:

$$\frac{x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; \textit{lub-E} \qquad \frac{z \leqslant w, x \leqslant w, y \leqslant w, \Gamma \Rightarrow \Delta}{x \leqslant w, y \leqslant w, \Gamma \Rightarrow \Delta}\; \textit{lub-U}$$

Rule *lub-E* has the condition that z is not in Γ, Δ, whereas rule *lub-U* has the condition that in a derivation it should always be applied above (but not necessarily immediately above) rule *lub-E* .

System of rules for generalized geometric implications

Case $n = 1$:

$$
\dfrac{
\begin{array}{c}
\Gamma'_1 \Rightarrow \Delta'_1 \\
\vdots \\
\mathcal{D}_0^1 \\
\vdots \\
\Gamma''_1 \Rightarrow \Delta''_1 \\
\vdots \\
\mathcal{D}^1 \\
\vdots \\
z_1 = z_1, \overline{P}, \Gamma \Rightarrow \Delta
\end{array}
\quad \dots \quad
\begin{array}{c}
\Gamma'_m \Rightarrow \Delta'_m \\
\vdots \\
\mathcal{D}_0^m \\
\vdots \\
\Gamma''_m \Rightarrow \Delta''_m \\
\vdots \\
\mathcal{D}^m \\
\vdots \\
z_m = z_m, \overline{P}, \Gamma \Rightarrow \Delta
\end{array}
}{
\overline{P}, \Gamma \Rightarrow \Delta
}
$$

where z_i are eigenvariables in the last inference step, the derivations indicated with $\mathcal{D}_0^i$ use rules of the form $GRS_0(z_i)$ that correspond to the geometric axioms $GA_0(z_i)$ in addition to logical rules, and the $\mathcal{D}^i$ use only logical rules.

System of rules for generalized geometric implications

The scheme GRS_{n+1} is defined inductively with the same conditions as above once the schemes GRS_n have been defined as follows

$$
\dfrac{
\begin{array}{c}
\Gamma'_1 \Rightarrow \Delta'_1 \\
\vdots \\
\mathcal{D}^1_n \\
\vdots \\
\Gamma''_1 \Rightarrow \Delta''_1 \\
\vdots \\
\mathcal{D}^1 \\
\vdots \\
z_1 = z_1, \overline{P}, \Gamma \Rightarrow \Delta
\end{array}
\quad \dots \quad
\begin{array}{c}
\Gamma'_m \Rightarrow \Delta'_m \\
\vdots \\
\mathcal{D}^m_n \\
\vdots \\
\Gamma''_m \Rightarrow \Delta''_m \\
\vdots \\
\mathcal{D}^m \\
\vdots \\
z_m = z_m, \overline{P}, \Gamma \Rightarrow \Delta
\end{array}
}{
\overline{P}, \Gamma \Rightarrow \Delta
}
$$

Other examples

- *Continuity axiom* $\forall\epsilon\exists\delta(\forall x(x \in B(\delta) \supset f(x) \in B(\epsilon)))$ is in GA_1.
- $(P \supset Q) \vee (Q \supset P)$ is in GA_1 (a degenerate case without variables).

  The system of rules has the form

$$\dfrac{\dfrac{\dfrac{Q, P, \Gamma' \Rightarrow \Delta'}{P, \Gamma' \Rightarrow \Delta'}}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{P, Q, \Gamma'' \Rightarrow \Delta''}{Q, \Gamma'' \Rightarrow \Delta''}}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

Equivalence of axiomatic systems and systems of rules

Using the rules that belong to the schema $GRS_0(z_i)$ we can derive $\& GA_0(z_i)$ and conversely, a derivation that uses the schema $GRS_0(z_i)$ can be converted into a purely logical derivation from the same assumptions, with $\& GA_0(z_i)$ added in the antecedent of the conclusion, that is:

For all n, **G3c** (**G3im**)$+GRS_n =$ **G3c**(**G3im**) $+ GA_n +$ cut

By induction on n; the base case, for geometric axioms, was proved in (N 2003).

Admissibility of structural rules. For all n, all the structural rules are admissible in **G3c** (**G3im**)$+GRS_n$.

Generalized first-order Barr's theorem

Barr's theorem: If a geometric implication is derivable classicaly in a geometric theory, then it is dervable inctuitionistically.

Generalized geometric implications clearly do not fall under the possible extensions of Barr's theorem. A counterexample is $(A \supset B) \vee (B \supset A)$ (with an empty theory). This is in GA_1 and has a level 1 disjunction.

Generalized Barr's theorem

For all n, if a geometric implication is derivable in **G3c** $+GRS_n$, it is derivable in **G3im** $+GRS_n$.

Proof: *Nothing to prove*. Derivation uses only rules from GRS_n and logical rules. Because of the shape of the conclusion, none of the rules that violates the intuitionistic single-succedent restrictions (i.e., the classical multisuccedent $R\supset$, $R\forall$) can have been used, so the derivation is already an intuitionistic one.

Characterization of generalized geometric implications in terms of Glivenko classes

Generalized geometric implications do not contain negative occurrences of implications ($\supset^-$) nor of universal quantifiers ($\forall^-$).

Any first-order formula A that does not contain $\supset^-$ or $\forall^-$ is intuitionistically (and actually minimally) equivalent to a conjunction of generalized geometric implications and can be converted to equivalent generalized geometric rules.

Operative conversion also gives the systems of rules equivalent to formula A.

Systems of rules for non-classical logics

Systems of rules can be used to obtain labelled proof systems for logics characterized in terms of relational semantics. A completeness theorem similar for the one for extensions with generalized geometric rules:

Soundness

If the sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3K}^*$, then it is valid in every frame with properties $*$ ranging over generalized geometric axioms.

Completeness

Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of $\mathbf{G3K}^*$. Then either the sequent is derivable in $\mathbf{G3K}^*$ or it has a Kripke countermodel with properties $*$.

The Sahlqvist fragment

Characterization of frame properties that correspond to Sahlqvist formulas

Kracht formulas

1. No variable is both free and bound and no two distinct occurrences of quantifiers bind the same formula.
2. *Restrictedly positive*, i.e., built from atomic formulas and negation of atomic formulas using only &, $\vee$, and *restricted quantifiers* Q^r (of the form $\forall^r yA(y) \equiv \forall y(xRy \supset A(y))$ and $\exists^r yA(y) \equiv \exists y(xRy \& A(y))$.
3. Every atomic subformula is either of the form $z = z$ or $z \neq z$, or else it contains at least one *inherently universal variable* (either free or bound by a restricted universal quantifier which is not in the scope of an existential quantifier).

Conversion to normal form for Kracht formulas

Conjunctions of

$$\forall^r x_1 \ldots \forall^r x_n Q_1^r y_1 \ldots Q_m^r y_m A(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

A is quantifier free; its atomic subformulas are either of the form $z = z$, or $z \neq z$ or contain one of the variables x_i.

The Sahlqvist fragment (cont.)

By the characterization of generalized geometric implications in terms of the Glivenko class $(\supset^-, \forall^-)$ we have:

Every Kracht formula is equivalent to a conjunction of generalized geometric implications.

and therefore

Generalized geometric rules provide complete proof systems for for any modal logic axiomatized by Sahlqvist formulas.

Conclusion

We have shown how contraction-free labelled sequent calculi unify the search of proofs and countermodels for non-classical logics. We have also shown several methods to obtain an *effective* unification:

- Search for minimal proofs
- Syntactic filtration
- Embedding into provability logic

Open questions

- Minimality of countermodels
- Syntactic embedding into **GL**?
- Most general scope of the method?
- Intuitionistic modal logics with possibility modalities?
- Decidability through proof search for systems with generalized geometric axioms?

References

- Dyckhoff, R. and S. Negri (2012) Proof analysis in intermediate logics. *Archive for Math. Logic*, vol. 51, pp. 71–92.
- Dyckhoff, R. and S. Negri (2013) A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel-McKinsey-Tarski embedding. J. of Logic and Computation, in press.
- Garg, G., V. Genovese, and S. Negri (2012) Counter-models from sequent calculi in multi-modal logics. *LICS 2012*, IEEE Computer Society, pp. 315–324.
- Maffezioli, P. , A. Naibo, and S. Negri (2012) The Church-Fitch knowability paradox in the light of structural proof theory. *Synthese*, Online first DOI 10.1007/s11229-012-0061-7
- Negri S. (2005) Proof analysis in modal logic. *Journal of Philosophical Logic*, vol. 34, pp. 507–544.
- Negri S. (2009) Kripke completeness revisited. In G. Primiero and S. Rahman (eds.), *Acts of Knowledge - History, Philosophy and Logic*, pp. 247–282, College Publications.
- Negri, S. (2013) Proof analysis beyond geometric theories: from rule systems to systems of rules, *Journal of Logic and Computation*, forthcoming.
- Negri, S. and J. von Plato (2011) *Proof Analysis*. Cambridge UP.
- Negri, S. and J. von Plato (2013) Meaning in use. In H. Wansing (ed) *Dag Prawitz on Proofs and Meaning*, Trends in Logic, Springer, in press.