Model Checking for Compositional Models of General Linear Time

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Introduction

- Real Temporal Logic: Simple, Expressively complete re: First Order Monadic Logic of Order.
- French et al. [2012] proposed a language of model expressions, based on pioneering work of Läuchli and Leonard [1966], Burgess and Gurevich [1985], Reynolds [2001, 2010]
  - Complete: Every satisfiable formula has a model in this language.
- Have complete and finite representations of models for Real Temporal Logic formulas
  - Obvious question: Can we Model Check?
  - Yes, and generalise to General Linear Time
  - And we have an efficient implementation [McCabe-Dansted, 2012]

Compositional Model Expressions

Model Expressions are an abstract syntax to define models using the following set of primitive operators based on the four operations:

$$\mathcal{I} ::= \; a \mid \lambda \mid \mathcal{I} + \mathcal{J} \mid \overleftarrow{\mathcal{I}} \mid \overrightarrow{\mathcal{I}} \mid \langle \mathcal{I}_0, \ldots, \mathcal{I}_n \rangle$$

where $a \in \Sigma = 2^{\mathbf{L}}$ so the letter indicates the atoms true at a point. We refer to these operators, respectively, as a letter, the empty order, concatenation, lead, trail, and shuffle.

- How does this relate to a model like $(\mathbb{Q}, <, g)$?

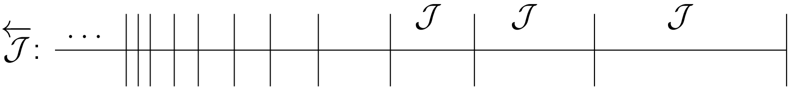
Correspondence

We define whether a model expression corresponds to a model recursively as follows: (or more formally in paper)

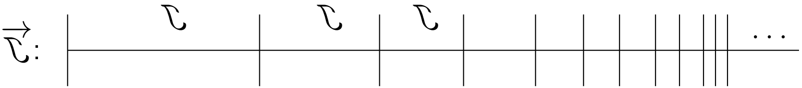
- A letter a corresponds to a single point at which the set of atoms satisfied is a.
- The empty order λ corresponds to an empty (psuedo-frame)
- The concatenation $\mathcal{I} + \mathcal{J}$ corresponds to a model where first $\mathcal{I}$, then $\mathcal{J}$.
- The lead $\overleftarrow{\mathcal{I}}$ corresponds to an infinite number of repeats of $\mathcal{I}$, forming a limit point on the left.
- The trail $\overrightarrow{\mathcal{I}}$ corresponds to an infinite number of repeats of $\mathcal{I}$, forming a limit point on the right.
- A shuffle $\langle \mathcal{I}_0, \ldots, \mathcal{I}_n \rangle$ corresponds to a dense mixture of $\mathcal{I}_0, \ldots, \mathcal{I}_n$.

Lead: $\mathcal{I} = \overleftarrow{\mathcal{J}}$

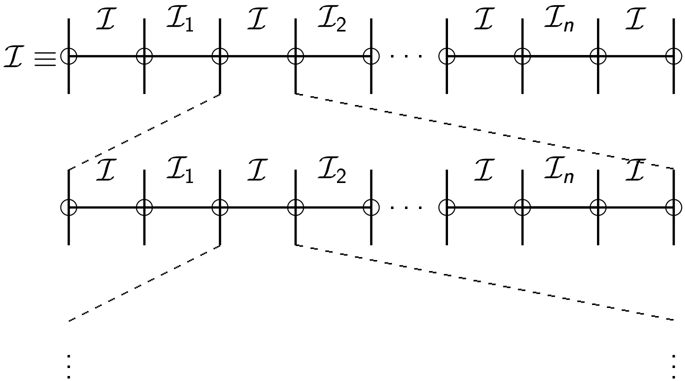
Here is a lead $\overleftarrow{\mathcal{J}}$.



A trail $\overrightarrow{\reflectbox{$\mathcal{J}$}}$ is just the mirror image of $\overleftarrow{\mathcal{J}}$.



Shuffle $\mathcal{I} = \langle \mathcal{I}_1, \ldots, \mathcal{I}_n \rangle$

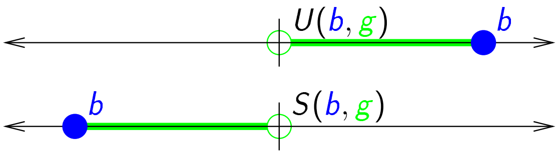


Language of Until and Since

- The Language of Until and Since

$$\phi ::= \; p \mid \neg\phi \mid \phi \wedge \phi \mid U(p,q) \mid S(p,q)$$

- In RTL:



- The US/L Logic like RTL, but models can be over any linear flow of time (not just Reals, e.g. Integers).

An Example: Zeno

If we walk into a wall: we will halve the distance; whenever we have halved the distance, we will halve the distance again, but only after a period of not halving the distance; once we have reached the wall we will not halve the distance anymore; and finally, we reach the wall. Where h represents "we have halved the distance" and r represents "we have reached the wall". The US/L formula for the intended expression is:

$$Fh \wedge G\left(h \rightarrow U\left(h, \neg h\right)\right) \wedge G\left(r \rightarrow G \neg h\right) \wedge Fr \ .$$

We see that this does not cause a contradiction as this formula is satisfied at the leftmost point x of any structure $\mathbf{T}$ corresponding to $\overrightarrow{\{h\} + \emptyset} + \{r\}$.

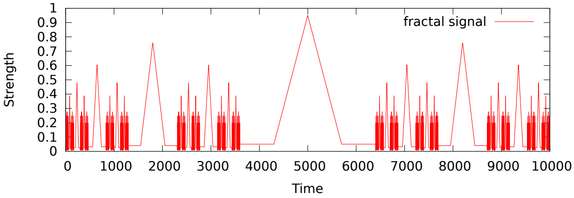
An Example: Detector

At some sporting event, it may be that a player is awarded a point if the ball bounces twice. An automated system may detect such bounces and make a ruling as to whether to award a point. The system may poll a given sensor, to determine whether a bounce has occurred since the previous polling event. If the system detects two bounces it awards a point. Where b indicates that a bounce has just occurred, s indicates that system has just checked its sensor, and e indicates the end of the round.

The player deserves a point precisely if $\theta = F(b \wedge F(b \wedge Fe))$ holds while the system awards a point precisely where the formula $\theta_s = F(b \wedge F(s \wedge F(b \wedge F(s \wedge Fe))))$ holds. The result is correct if $\theta \leftrightarrow \theta_s$ holds. We can verify the result is correct for a run of the system against the environment described by the ME:

$$\{s\} + \{b\} + \{s\} + \{s\} + \{b\} + \{s\} + \overrightarrow{\{b\}} + \{e\}$$

An Example: Fractal Signal



Let p represent an increment, q describe a decrement, and r represent a constant signal, then we can model this signal using:

$$\langle\langle r\rangle + \langle p\rangle + \langle q\rangle + \langle r\rangle\rangle$$

Some properties of this signal can be represented in $L(U, S)$, for example we can reach a region of increment directly from a constant region so $r \wedge U(p, r \vee p)$ is satisfied within the model. However, we cannot reach a region of increment directly from a region of decrement, so $q \wedge U(p, q)$ is satisfied nowhere.

Model Checking

Definition

We define the model checking problem as follows: given an ME $\mathcal{I}$ and formula ϕ, determine whether there exists a structure $\mathbf{T} = (T, <, h)$ corresponded to by $\mathcal{I}$ and point $x \in T$ such that $\mathbf{T}, x \vDash \phi$.

A traditional approach to model checking is to add subformula as atoms

Definition

The model checking procedure takes as input an ME $\mathcal{I}$ and formula ϕ. We enumerate the subformulas $\phi_1, \ldots, \phi_n$ of ϕ from shortest to longest (so $\phi_n = \phi$), let $\mathcal{I}_0 = \mathcal{I}$, and $\mathcal{I}_i = \text{add\_atom}_{\phi_i}(\mathcal{I}_{i-1})$ for each $i \in \{0, \ldots, n\}$. Finally, we return "true" if there is a letter a in $\mathcal{I}_n$ such that $\phi \in a$, and "false" otherwise.

Adding $\wedge$ and $\neg$ as atoms

Adding PC formulas as atoms is trivial consider:

	ϕ	$\mathcal{I}$
1	$(a \wedge b) \wedge \neg c$	$\{a, b\} + \{c\}$
2	$(a \wedge b) \wedge p_{\neg c}$	$\{a, b, p_{\neg c}\} + \{c\}$
3	$p_{a \wedge b} \wedge p_{\neg c}$	$\{a, b, p_{a \wedge b}, p_{\neg c}\} + \{c\}$
4	$p_{(a \wedge b) \wedge \neg c}$	$\{a, b, p_{a \wedge b}, p_{\neg c}, p_{(a \wedge b) \wedge \neg c}\} + \{c\}$

Pre(satisfaction)

"pre $(\mathcal{K}, \dashv)$" is true if $U(p, q)$ is true before $\mathcal{K}$ where

- $\mathcal{K}$ is an ME
- And $\dashv$ means "$U(p, q)$" is true after $\mathcal{K}$

Definition

We define a function "pre" from Booleans and MEs to Booleans such that: for any Boolean $\dashv$ and pair of MEs $\mathcal{I}, \mathcal{J}$

1. $\text{pre}\,(a, \dashv) = p \in a \vee (\dashv \wedge q \in a)$
2. $\text{pre}\,(\mathcal{I} + \mathcal{J}, \dashv) = \text{pre}\,(\mathcal{I}, \text{pre}\,(\mathcal{J}, \dashv))$
3. $\text{pre}\left(\overrightarrow{\mathcal{I}}, \dashv\right) = \text{pre}\,(\mathcal{I}, \dashv) \quad = \text{pre}\,(\mathcal{I}, \text{pre}\,(\mathcal{I}, \dashv))$
4. $\text{pre}\,(\mathcal{J}, \dashv) = (\dashv \vee \exists l \in L\,(\mathcal{J})$ s.t. $p \in l) \wedge \forall l \in L\,(\mathcal{J})\,,\ q \in l$; where $\mathcal{J}$ is of the form $\overleftarrow{\mathcal{I}}$ or $\langle \ldots \rangle$ and $L\,(\mathcal{J})$ is the set of letters within $\mathcal{J}$.

Adding Until as an Atom

Definition

We define $\text{add\_atom}_{U(p,q)}(\mathcal{I})$ as $t(\mathcal{I}, \bot)$: where t is a function that takes an ME and a Boolean as input, and outputs an ME as follows: for any Boolean $\dashv$, pair of MEs $\mathcal{I}, \mathcal{J}$ and sequence of MEs $\mathcal{I}_0, \ldots, \mathcal{I}_n$

1. $t(a, \dashv) = \begin{cases} a & \text{if } \dashv = \bot \\ a \cup \{U(p, q)\} & \text{if } \dashv = \top \end{cases}$
2. $t(\mathcal{I} + \mathcal{J}, \dashv) = t(\mathcal{I}, \text{pre}(\mathcal{J}, \dashv)) + t(\mathcal{J}, \dashv)$
3. $t\left(\overleftarrow{\mathcal{I}}, \dashv\right) = \overleftarrow{t(\mathcal{I}, \text{pre}(\mathcal{I}, \dashv))} + t(\mathcal{I}, \dashv)$
4. $t\left(\overrightarrow{\mathcal{I}}, \dashv\right) = \overrightarrow{t(\mathcal{I}, \text{pre}(\mathcal{I}, \dashv))}$
5. $t(\mathcal{K}, \dashv) = \langle t(\mathcal{I}_0, \dashv'), \ldots, t(\mathcal{I}_n, \dashv')\rangle$ where $\mathcal{K} = \langle \mathcal{I}_0, \ldots, \mathcal{I}_n\rangle$ and $\dashv' = \text{pre}(\mathcal{K}, \dashv)$

Complexity (1)

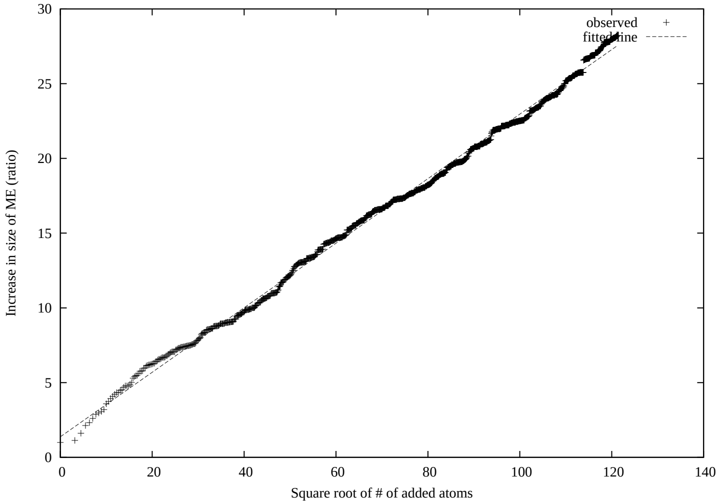
- Adding one Until: $\overleftarrow{\mathcal{I}} \rightsquigarrow \overleftarrow{\mathcal{I}_0} + \mathcal{I}_1$
- Adding n Untils: $\overleftarrow{\mathcal{I}} \rightsquigarrow \overleftarrow{\mathcal{I}_0} + \mathcal{I}_1 + \cdots + \mathcal{I}_n$
- With m nested leads, has $\approx (n+1)^m$ length
- $\mathcal{O}\left(|\phi|^{|\mathcal{I}|}\right)$
- **Polynomial in Length of the Formula**

Complexity (2)

Traditionally we want model checking to be polynomial in the Model

- Represent MEs as Directed Acyclic Graphs (1 node per unique subME)
- Each unique $\mathcal{I}$ becomes $t(\mathcal{I}, \top)$ or $t(\mathcal{I}, \bot)$
- Adding U as atom ~~doubles~~ **triples** # unique nodes
    - E.g. $\overleftarrow{\{p\}}$ becomes $\overleftarrow{\{p, U(p, p)\}} + \{p\}$
- $|\mathcal{I}|\, 3^{|\phi|}$ nodes (or $|\mathcal{I}|\, |\phi|\, 2^{|\phi|}$)
- **Linear in Length of ME**

Benchmark: Random Square Problem(s)



Plot of growth of $|\mathcal{I}_n|/|\mathcal{I}_0|$ vs. $\sqrt{n}$ for $2^{15} \times 2^{15}$ problem

Estimated Space: $\approx |\mathcal{I}|\,|\phi|^{3/2}$ and Time $\approx |\mathcal{I}|\,|\phi|^{5/2}$

Ongoing Work

Every RTL model M of an formula ϕ is also a US/L model.

- Can model check Real MEs using same model checker
- Minor Detail: MEs correspond to <u>countable</u> structures

Expressive completeness re Linear flows requires Stavi Until/Since

- Can translate into RTL using special atom c for gaps.

Can translate Metrics with error into RTL.

Conclusions

- ME provide models for all RTL formulas.
- Found efficient model checker:
  - Polynomial in Length of Formula
  - Linear in length of ME
  - Moderate $\sqrt{n}$ growth on random formulas
  - But PSPACE-Complete in general (Present at TIME 2013)
- Extensions for Metrics, Stavi etc. inprogress.

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NP-Hardness (PSPACE-Hardness)

We can show NP-Hardness (extend to PSPACE-hardness in Paper)

1. $\mathcal{I}_1 = \overleftarrow{p_0} + q_0$
2. $\mathcal{I}_i = \overleftarrow{p_i + \mathcal{I}_{i-1}} + q_i$

SAT problem: Is $\phi = (r_1 \wedge r_0) \vee \neg r_0$ satisfiable

- replace r_i with $r_i' = U(q_i, \neg p_i)$
    - i.e. $\phi' = (U(q_1, \neg p_1) \wedge U(q_0, \neg p_0)) \vee \neg U(q_0, \neg p_0)$
- if n atoms in ϕ: model check $\mathcal{I}_n$ against ϕ'

Example

- Model check against $\mathcal{I}_1$ (or $\mathcal{I}_{n-1}$ if n atoms in ϕ)
- $\mathcal{I}_2 = \overleftarrow{p_1 + \overleftarrow{p_0} + q_0} + q_1$
- $\mathcal{I}_2 \rightsquigarrow \overleftarrow{p_1 + \overleftarrow{\{p_0\}} + \{p_0, U(q_0, \neg p_0)\} + q_0} + q_1$
- $\mathcal{I}_2 \rightsquigarrow \overleftarrow{p_1 + \overleftarrow{\{p_0\}} + \{p_0, U(q_0, \neg p_0)\} + q_0} +$ $\overleftarrow{\{p_0, U(q_1, \neg p_1)\}} + \{p_0, U(q_0, \neg p_0), U(q_1, \neg p_1)\} + q_1$

PSPACE

We use a recursively defined function $\mathfrak{A}$. Informally, $\mathfrak{A}(\mathcal{I}, \Phi, \phi) = (\Theta, \Psi)$ means

- if we have an interval $\mathbf{T}_V$ that corresponds to $\mathcal{I}$
  - Φ is the set of formulas in $\mathbb{U}$ presatisfied immediately after V in $\mathbf{T}$
  - we are only interested in ϕ and subformulas of ϕ,
- Then it must be the case that the set of formulas satisfied within $\mathbf{T}_V$ is Θ,
  - and the set formulas in $\mathbb{U}$ presatisfied immediately before V in $\mathbf{T}$ is Ψ.
  - The formula ϕ indicates that we are only interested in whether $\phi \in \Theta$ and so we can limit ourselves to subformulas of ϕ and their negations. The algorithm works by generating increasing accurate approximations to (Θ, Ψ) that are accurate up to some subformula ϕ_j.

Excerpt of Recursive Function $\mathfrak{A}$

Definition

Let $\mathcal{K}$ be an ME, and Φ be a set of formulae. We define $\mathfrak{A}(\mathcal{K}, \Phi, \phi)$ to be the pair of sets of formulas (Θ, Ψ) as follows.

We consider various possible forms of $\mathcal{K}$. The first case we consider is a letter. In the following construction we build increasingly accurate approximations of (Θ, Ψ): for each j we have $\Theta_j \approx_{\leq j} \Theta$ and $\Psi_j \approx_{\leq j} \Psi$.

Case 0. $\mathcal{K} = \lambda$. Since $\mathcal{K}$ corresponds to the empty pseudo frame it cannot satisfy any formula so we let $\Theta = \emptyset$, likewise it cannot affect pre or postsatisfaction so $\Psi = \Phi$.

Case 1. $\mathcal{K}$ is a letter:

- $\Theta_0 = \emptyset$, $\Psi_0 = \emptyset$
- for each i:
  - We let Θ_i be a set such that $\Theta_i \setminus \{\phi_i\} = \Theta_{i-1} \setminus \{\phi_i\}$ and: if ϕ_i is of the form $\neg\alpha$ then $\phi_i \in \Theta_i$ iff $\alpha \notin \Theta_{i-1}$; if ϕ_i is of the form $\alpha \wedge \beta$ then $\phi_i \in \Theta_i$ iff $\alpha \in \Theta_{i-1} \wedge \beta \in \Theta_{i-1}$; if ϕ_i is an atom then $\phi_i \in \Theta_i$ iff $\phi_i \in \mathcal{K}$; if $\phi_i \in \mathbb{U} \cup \mathbb{S}$ then $\phi_i \in \Theta_i$ iff $\phi \in \Phi$.
  - $\phi_i \in \Psi$ iff ϕ_i is of the form $U(\alpha, \beta)$ or $S(\alpha, \beta)$ and either $\alpha \in \Theta_i$ or $(\beta \in \Theta_i) \wedge (\phi_i \in \Phi)$.
  - We let Θ be the minimal expansion of Θ_n such that for each ϕ_i, we have $\phi_i \in \Theta$ iff $\phi_i \in \Theta_n$ and $\neg\phi_i \in \Theta$ iff $\phi_i \notin \Theta_n$. We add the negations into Θ so that when we have an ME with multiple letters we can express "α occurs everywhere" as $\neg\alpha \notin \Theta$.

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