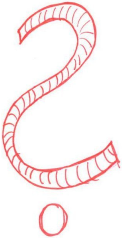
Hypersequent and Labelled Calculi for Intermediate Logics

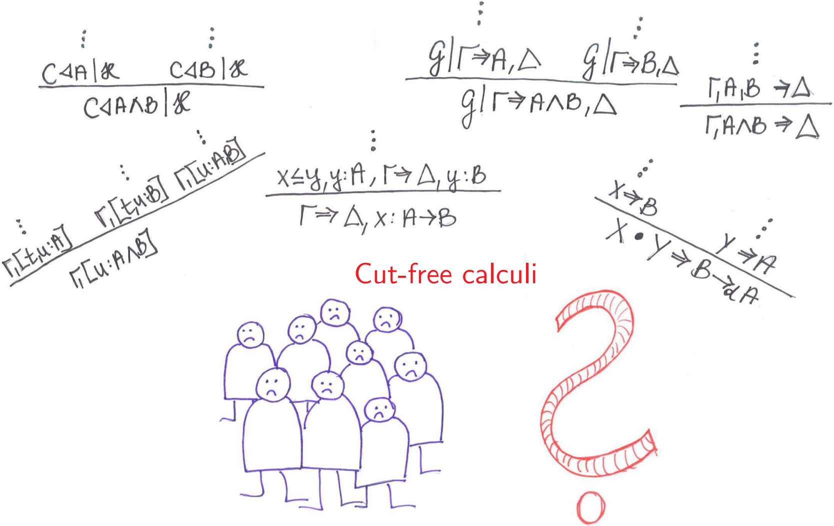
Lara Spendier
(Joint work with A. Ciabattoni and P. Maffezioli)

lara@logic.at
Vienna University of Technology

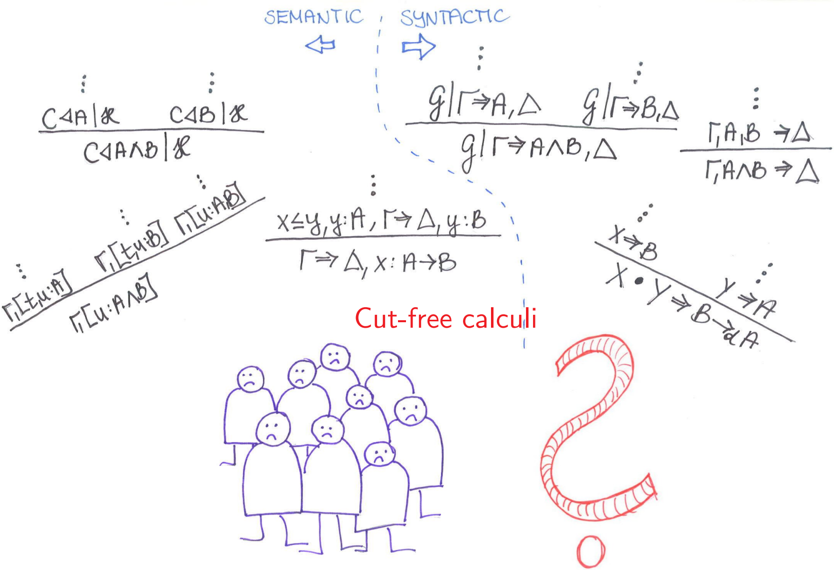
September 19, 2013

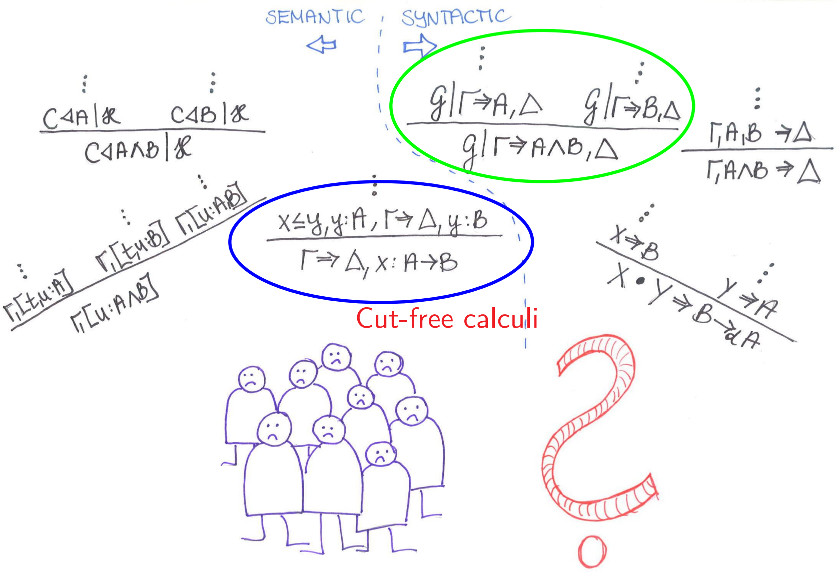
Cut-free calculi





Cut-free calculi


SEMANTIC
SYNTACTIC
Cut-free calculi



Cut-free calculi

Intermediate logics: Two approaches

Intermediate logics ...

- Logics between intuitionistic and classical logic

Semantic approach

- imposing on intuitionistic Kripke frames additional conditions on the (transitive and reflexive) accessibility relation $\leqslant$
- Labelled calculi

Syntactic approach

- extending intuitionistic logic with Hilbert axioms
- Hypersequent calculi

(Propositional) Intermediate logics

Example

Jankov (De Morgan) logic: IL +

Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to \exists w(y \leqslant w \wedge z \leqslant w))$
"Equivalent" axiom	$\neg\alpha \vee \neg\neg\alpha$

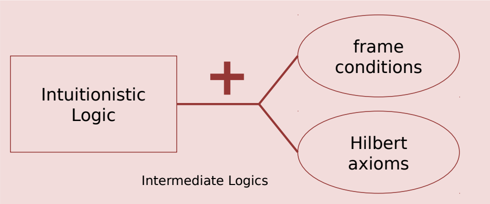
Gödel logic: IL +

Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to (y \leqslant z \vee z \leqslant y))$
"Equivalent" axiom	$(\alpha \supset \beta) \vee (\beta \supset \alpha)$

Bd_2: IL +

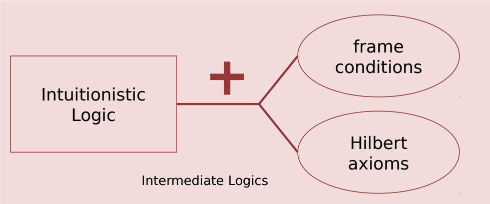
Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge y \leqslant z) \to (y \leqslant x \vee z \leqslant y))$
"Equivalent" axiom	$\alpha \vee (\alpha \supset (\beta \vee \neg\beta))$

Intermediate logics: Two approaches

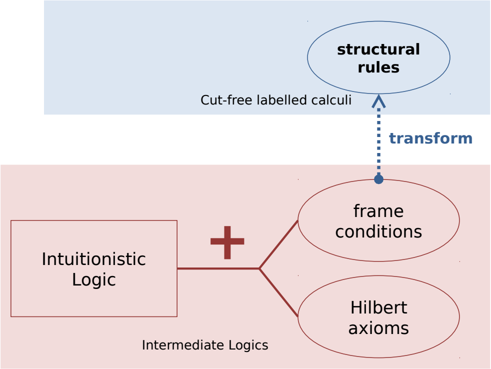


Intermediate logics: Two approaches

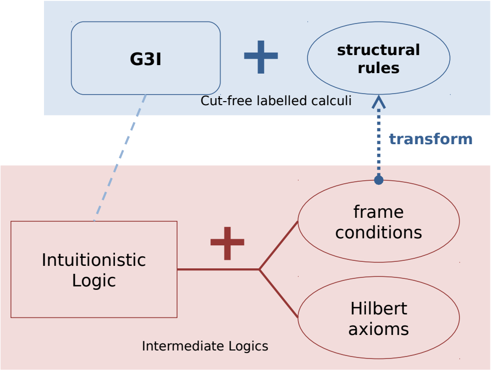
Cut-free labelled calculi



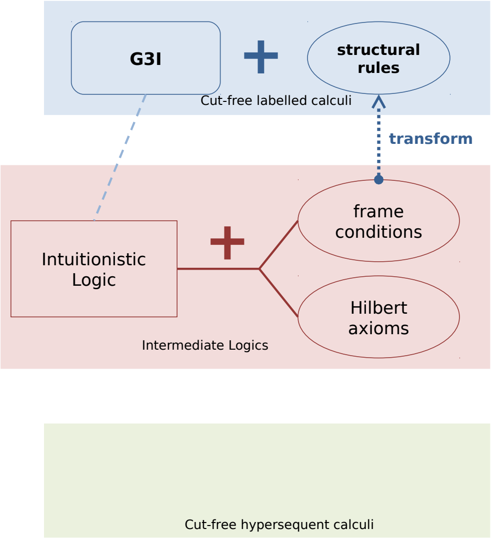
Intermediate logics: Two approaches



Intermediate logics: Two approaches



Intermediate logics: Two approaches



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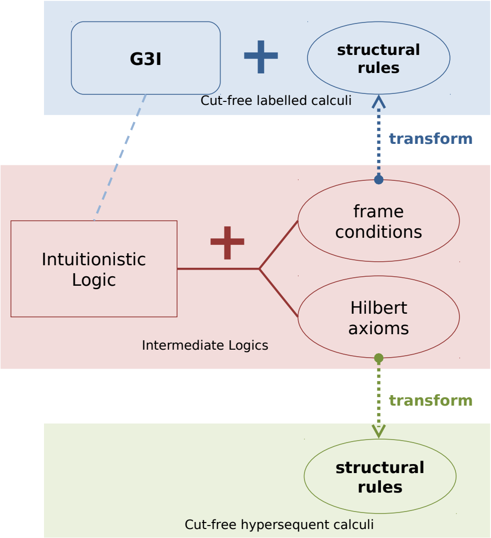

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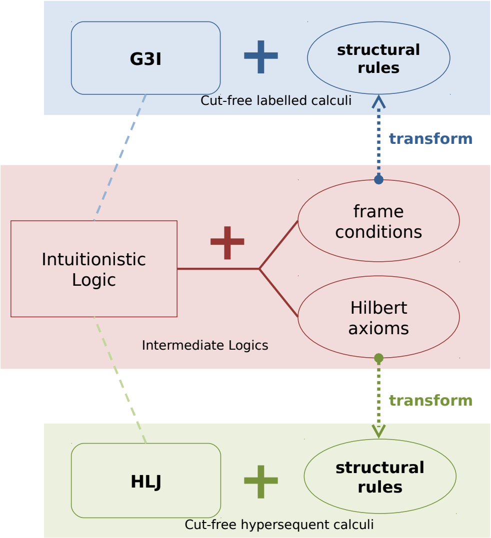
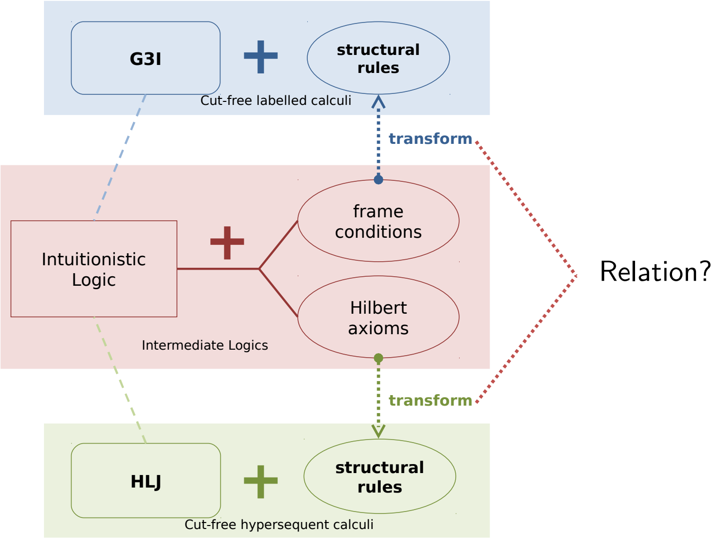
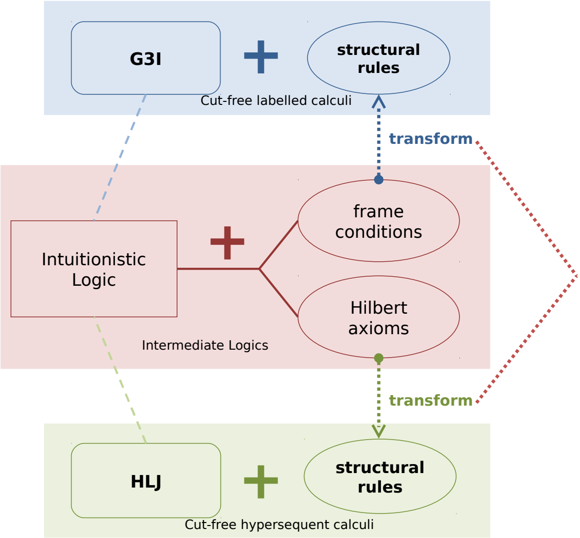


Intermediate logics: Two approaches



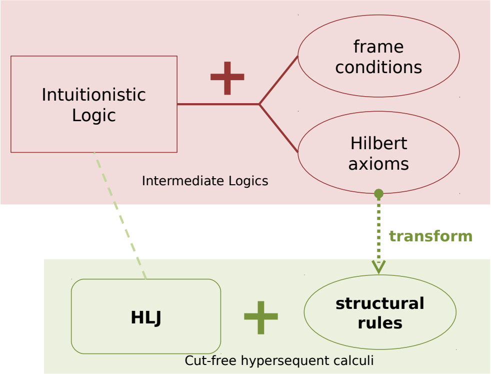
Intermediate logics: Two approaches



Relation?

- compare
- extend
- systematise

Hypersequent calculi & intermediate logics



Hypersequent calculi: State of the art

A. Ciabattoni, N. Galatos and K. Terui, 2008:

(Step 1): Classification of axioms based on:

Polarity of connectives (J.-M. Andreoli, 1992)

(Step 2): Transformation procedure based on 2 key concepts:

1. Invertibility of rules
2. Ackermann Lemma

Hypersequent calculi: Step 1 – Classification

- *Positive* polarity: rule introducing the connective on the *left* is invertible
- *Negative* polarity: rule introducing the connective on the *right* is invertible

Example (LJ)

$$\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\ (\wedge, l) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge, r)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \vee \beta, \Gamma \Rightarrow \delta}\ (\vee, l) \qquad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee_i, r)$$

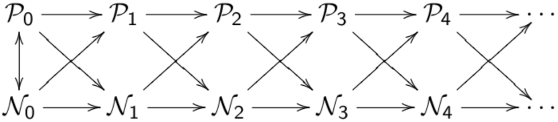
$$\frac{\Gamma \Rightarrow \alpha \qquad \beta, \Gamma \Rightarrow \delta}{\alpha \supset \beta, \Gamma \Rightarrow \delta}\ (\supset, l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \supset \beta}\ (\supset, r)$$

Hypersequent calculi: Step 1 – Classification

Definition (Classification LJ; A. Ciabattoni et al., 2008)

The classes $\mathcal{P}_n, \mathcal{N}_n$ are defined as follows:

- $\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ set of atomic formulas
- $\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \bot$
- $\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \top$



Hypersequent calculi: Step 2 – Transformation procedure

(Step 1): Classification of formulas based on:

Polarity of connectives (J.-M. Andreoli, 1992)

(Step 2): Transformation procedure based on 2 key concepts:

1 *Invertibility of rules*

- Invertible rules applied as often as possible
- We can get stuck:

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- Invertible rules applied as often as possible
- We can get stuck:

Example (Formula $\alpha \supset \beta \vee \gamma$)

$$\frac{}{G \mid \Rightarrow \alpha \supset \beta \vee \gamma} \quad \text{Invertibility } (\supset, r)$$

$$\frac{}{G \mid \alpha \Rightarrow \beta \vee \gamma} \quad \text{Stuck!}$$

Hypersequent calculi: Step 2 – Transformation procedure

(Step 1): Classification of formulas based on:

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(Step 2): Transformation procedure based on 2 key concepts:

1 Invertibility of rules

2 *Ackermann Lemma*

- Move formulas from the conclusion of (hyper)sequent rules to their premises by making them change the (hyper)sequent side.

Hypersequent calculi: Step 2 – Transformation procedure

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2. *Ackermann Lemma*
   - Move formulas from the conclusion of (hyper)sequent rules to their premises by making them change the (hyper)sequent side.

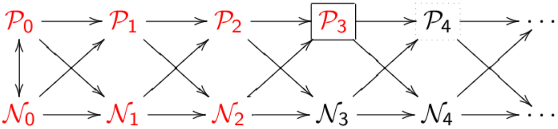
Example (Formula $\alpha \supset \beta \vee \gamma$)

$$\frac{}{G \mid \alpha \Rightarrow \beta \vee \gamma}$$ Ackermann Lemma

$$\frac{G \mid \beta \vee \gamma, \Gamma \Rightarrow \chi}{G \mid \alpha, \Gamma \Rightarrow \chi}$$ Invertibility $(\vee, l)$

Hypersequent calculi: From axioms to structural rules

Classification of Hilbert axioms:



Theorem (A. Ciabattoni et al., 2008)

An axiom belonging to a class up to $\mathcal{P}_3$ can be transformed into a set of equivalent structural hypersequent rules.

Theorem (A. Ciabattoni et al., 2008)

The generated rules added to the base calculus HLJ lead to a cut-free hypersequent calculus for the logic.

Limitations

- Transformation procedure for axioms belonging to $\mathcal{P}_3$
- $\mathcal{P}_3$ intuition: *(sequent)* $\vee$ *(sequent)* $\vee \cdots \vee$ *(sequent)*

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- $\mathcal{P}_3$ intuition: *(sequent)* $\vee$ *(sequent)* $\vee \cdots \vee$ *(sequent)*

Example

Jankov (De Morgan) logic: IL +

Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \rightarrow \exists w(y \leqslant w \wedge z \leqslant w))$
"Equivalent" axiom	$\neg\alpha \vee \neg\neg\alpha \in \mathcal{P}_3$

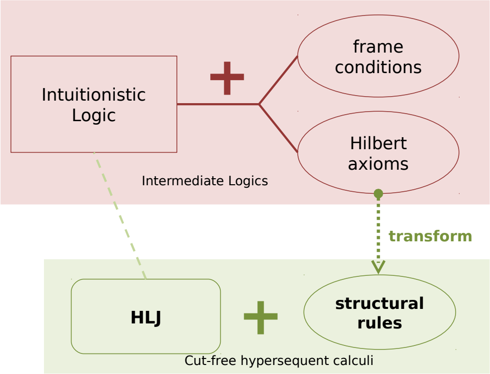
Gödel logic: IL +

Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \rightarrow (y \leqslant z \vee z \leqslant y))$
"Equivalent" axiom	$(\alpha \supset \beta) \vee (\beta \supset \alpha) \in \mathcal{P}_2$

Bd_2: IL +

Frame condition	$\forall x \forall y \forall z((x \leqslant y \wedge y \leqslant z) \rightarrow (y \leqslant x \vee z \leqslant y))$
"Equivalent" axiom	$\alpha \vee (\alpha \supset (\beta \vee \neg\beta)) \in \mathcal{P}_4$

First result: An extension



- extension:
  a *logical* rule

First result: A calculus for Bd_2 $\alpha \vee (\alpha \supset (\beta \vee \neg\beta))$

- Use the hypersequent version of Maehara's calculus for intuitionistic logic HLJ' (S. Maehara, 1954)
  - only in the right rule for $\supset$, the consequent of a sequent contains at most 1 formula

$$\frac{}{G \mid \alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{G \mid \bot \Rightarrow}\ (\bot, l) \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \alpha, \Delta}\ (w, r) \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (w, l)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha, \Delta \quad G \mid \Gamma, \beta \Rightarrow \Delta}{G \mid \Gamma, \alpha \supset \beta \Rightarrow \Delta}\ (\supset, l) \qquad \frac{G \mid \Gamma, \alpha \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow \alpha, \alpha, \Delta}{G \mid \Gamma \Rightarrow \alpha, \Delta}\ (c, r)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha, \Delta \quad G \mid \Gamma \Rightarrow \beta, \Delta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta, \Delta}\ (\wedge, r) \qquad \frac{G \mid \alpha, \beta, \Gamma \Rightarrow \Delta}{G \mid \alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge, l) \qquad \frac{G \mid \Gamma, \alpha, \alpha \Rightarrow \Delta}{G \mid \Gamma, \alpha \Rightarrow \Delta}\ (c, l)$$

$$\frac{G \mid \alpha, \Gamma \Rightarrow \Delta \quad G \mid \beta, \Gamma \Rightarrow \Delta}{G \mid \alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee, l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha, \beta, \Delta}{G \mid \Gamma \Rightarrow \alpha \vee \beta, \Delta}\ (\vee, r) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (ec)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha, \Delta \quad G \mid \alpha, \Sigma \Rightarrow \Pi}{G \mid \Gamma, \Sigma \Rightarrow \Pi, \Delta}\ (cut) \qquad \frac{G \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ (ee) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (ew)$$

First result: A calculus for Bd_2 $\alpha \vee (\alpha \supset (\beta \vee \neg \beta))$

- Use the hypersequent version of Maehara's calculus for intuitionistic logic HLJ' (S. Maehara, 1954)
- A heuristic method to create a *logical* rule and add it to HLJ':
  - Application of the transformation procedure to get a logical rule
  - Proving cut-elimination — fails
  - Analysing the counter-example gives the "right" logical rule

First result: A calculus for Bd_2 $\alpha \vee (\alpha \supset (\beta \vee \neg\beta))$

Logical rule for $(\supset, r)$ in IL:

$$\frac{G \mid \Gamma, \alpha \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta} \, (\supset, r)$$

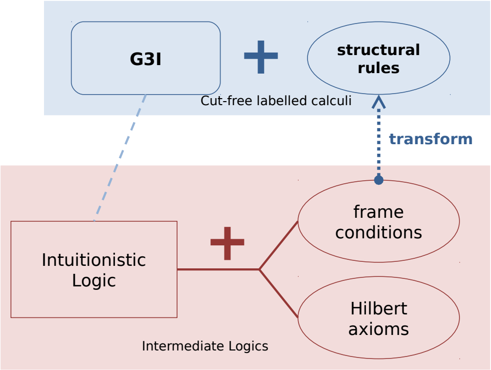
Logical rule for Bd_2:

$$\frac{G \mid \Gamma', \Gamma \Rightarrow \Delta' \qquad G \mid \Gamma, \alpha \Rightarrow \beta, \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \alpha \supset \beta, \Delta} \, (Bd_2^*)$$

Theorem

HLJ' extended with the rule (Bd_2^*) is a cut-free calculus for the logic Bd_2.

Labelled calculi & intermediate logics



Labelled calculi for intermediate logics

Labelled calculus *G3I* (S. Negri and J. von Plato, 2001):

$$x \leqslant y, x : p, \Gamma \Rightarrow \Delta, y : p$$

$$\frac{x : \alpha, x : \beta, \Gamma \Rightarrow \Delta}{x : \alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge, l) \qquad \frac{\Gamma \Rightarrow \Delta, x : \alpha \quad \Gamma \Rightarrow \Delta, x : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \wedge \beta}\ (\wedge, r)$$

$$\frac{x : \alpha, \Gamma \Rightarrow \Delta \quad x : \beta, \Gamma \Rightarrow \Delta}{x : \alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee, l) \qquad \frac{\Gamma \Rightarrow \Delta, x : \alpha, x : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \vee \beta}\ (\vee, r)$$

$$\frac{x \leqslant y, x : \alpha \supset \beta, \Gamma \Rightarrow \Delta, y : \alpha \quad x \leqslant y, x : \alpha \supset \beta, y : \beta, \Gamma \Rightarrow \Delta}{x \leqslant y, x : \alpha \supset \beta, \Gamma \Rightarrow \Delta}\ (\supset, l)$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ (\bot, l) \qquad \frac{x \leqslant y, y : \alpha, \Gamma \Rightarrow \Delta, y : \beta}{\Gamma \Rightarrow \Delta, x : \alpha \supset \beta}\ (\supset, r)$$

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (Ref) \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ (Trans)$$

Labelled calculi: State of the art

S. Negri, 2007, R. Dyckhoff and S. Negri, 2012:

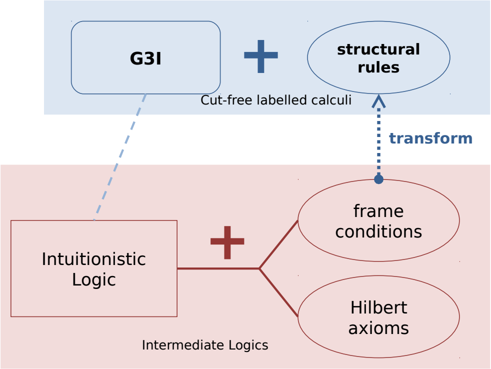
- Frame conditions following the "geometric axiom scheme" can be converted into structural rules that preserve cut-elimination when added to *G3I*.

$$\forall \overline{x}(\neg P_1 \vee \cdots \vee \neg P_m \vee \exists \overline{y}_1(\bigwedge Q_1) \vee \cdots \vee \exists \overline{y}_n(\bigwedge Q_n))$$

- Geometric rule scheme ($\overline{Q}_i, \overline{P}$ are sets of atoms, z_i are eigenvariables):

$$\frac{\overline{Q}_1[z_1/y_1], \overline{P}, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q}_k[z_k/y_k], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}$$

Second result: Towards a systematisation



- Why geometric axioms?

Second result: Towards a systematisation

(Step 1): Classification of axioms based on:

Polarity of connectives/quantifiers (J.-M. Andreoli, 1992)

(Step 2): Transformation procedure based on 2 key concepts:

1. Invertibility of rules
2. Ackermann Lemma

Second result: Step 1 – Classification

- Observation: Frame conditions are formulas of classical logic
- Polarities: $\exists$: positive – $\forall$: negative

Definition (Classification: arithmetical hierarchy)

Let A be a formula in prenex form. The classes Π_k, Σ_k are defined as follows:

- If A is quantifier-free then $A \in \Sigma_0$ and $A \in \Pi_0$.
- If A is classically equivalent to $\exists x B$ where $B \in \Pi_n$, then $A \in \Sigma_{n+1}$
- If A is classically equivalent to $\forall x B$ where $B \in \Sigma_n$, then $A \in \Pi_{n+1}$

Second result: Towards a systematisation

Example (Jankov logic)

Frame condition $\forall x \forall y \forall z((x \leqslant y \wedge x \leqslant z) \to \exists w(y \leqslant w \wedge z \leqslant w))$

Invertibility

$$\frac{}{\Rightarrow \forall x \forall y \forall z(x \leqslant y \wedge x \leqslant z) \to \exists w(y \leqslant w \wedge z \leqslant w)}$$

Ackermann Lemma

$$\frac{}{x' \leqslant y', x' \leqslant z' \Rightarrow \exists w(y' \leqslant w \wedge z' \leqslant w)}$$

Invertibility

$$\frac{\exists w(y' \leqslant w \wedge z' \leqslant w), \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Invertibility

$$\frac{y' \leqslant w' \wedge z' \leqslant w', \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Equivalent rule

$$\frac{y' \leqslant w', z' \leqslant w', \Gamma \Rightarrow \Delta}{x' \leqslant y', x' \leqslant z', \Gamma \Rightarrow \Delta}$$

Second result: Towards a systematisation

- Geometric axioms $\in \Pi_2$:

$$\forall \overline{x}(\neg P_1 \vee \cdots \vee \neg P_m \vee \exists \overline{y}_1 M_1 \vee \cdots \vee \exists \overline{y}_n M_n)$$

  where each P_i is an atom, each M_j is a conjunction of atomic formulas $Q_{j_1}, \ldots, Q_{j_k}$ and the variables y_j are not free in P_i.

- Any formula $\in \Pi_2$:

$$\forall \overline{x} \exists \overline{y} A$$

  where A is a quantifier-free formula.

Second result: Towards a systematisation

- Geometric rule scheme:

$$\frac{\overline{Q}_1[z_1/y_1], \overline{P}, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q}_k[z_k/y_k], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}$$

- Π_2 rule scheme:

$$\frac{\overline{Q}_1[z_1/y_1], \Gamma \Rightarrow \Delta, \overline{P_1} \quad \cdots \quad \overline{Q}_k[z_k/y_k], \Gamma \Rightarrow \Delta, \overline{P_k}}{\Gamma \Rightarrow \Delta}$$

($\overline{Q_i}, \overline{P}, \overline{P_i}$ sets of atoms, z_i eigenvariables)

Second result: Towards a systematisation

Theorem

A formula belonging to a class up to Π_2 can be transformed into a set of equivalent structural rules in labelled calculi.

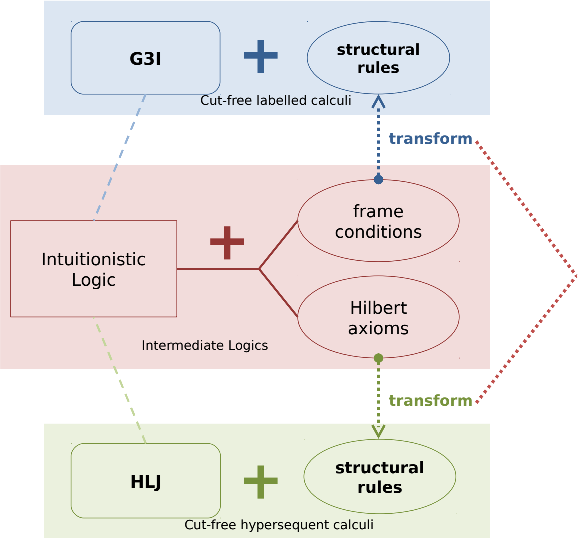
Theorem

Soundness, completeness and cut-elimination hold for the calculi obtained by extending G3I (+ the axiom $x \leq y, \Gamma \Rightarrow \Delta, x \leq y$) with the generated rules.

Corollary

The rules generated from geometric axioms by the transformation procedure are equivalent to the geometric rules from (Dyckhoff and Negri, 2012).

Conclusion



Relation:

- compare $\surd$
- extend $\surd$
- systematise $\surd$

Conclusion

We have:

- A method to turn a particular axiom into an equivalent hypersequent logical rule; thus, defining a first cut-free hypersequent calculus for the logic Bd_2

- A systematic method to generate labelled cut-free calculi for intermediate logics characterized by frame conditions that are Π_2 formulas, subsuming the class of geometric formulas.

Conclusion

We have:

- A method to turn a particular axiom into an equivalent hypersequent logical rule; thus, defining a first cut-free hypersequent calculus for the logic Bd_2

- A systematic method to generate labelled cut-free calculi for intermediate logics characterized by frame conditions that are Π_2 formulas, subsuming the class of geometric formulas.

We need help!

Conclusion: Questions

- A method to turn a particular axiom into an equivalent hypersequent logical rule; thus, defining a first cut-free hypersequent calculus for the logic Bd_2
  $\Rightarrow$ *How to generalise this method?*

- A systematic method to generate labelled cut-free calculi for intermediate logics characterized by frame conditions that are Π_2 formulas, subsuming the class of geometric formulas.
  $\Rightarrow$ *Are there frame conditions in Π_2 not equivalent to any geometric formula?*

- For Hilbert axioms: hierarchy collapses at level $\mathcal{N}_3$ (A. Chagrov and M. Zakharyaschev, 1997)
  $\Rightarrow$ For frame conditions above Π_2: *What is the maximum nesting of quantifiers?*