TAFA – A Tool for Admissibility in Finite Algebras

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TAFA

- Implemented in Delphi XE2
- Compiled for Windows (use WINE for Mac and Linux)
- Most recent version of TAFA.EXE is downloadable from `https://sites.google.com/site/admissibility/`

Derivability vs Admissibility

Consider a system defined by two rules:

$$Nat(0) \qquad \text{and} \qquad Nat(x) \Rightarrow Nat(s(x)).$$

The following rule is derivable:

$$Nat(x) \Rightarrow Nat(s(s(x))).$$

However, this rule is only admissible:

$$Nat(s(x)) \Rightarrow Nat(x).$$

But what if we add to the system:

$$Nat(s(-1)) \qquad ???$$

Motivation

- Admissibility plays a fundamental role in describing properties of (classes of) algebras and logics.
- Checking admissibility in finite algebras with the naive approach is decidable, but not feasible.
- We consider a more efficient method to check admissibility in finite algebras and provide a tool to get the results.

Validity and Admissibility

If $\Sigma \cup \{\varphi \approx \psi\}$ is a finite set of $\mathcal{L}$-equations, then we call the ordered pair $\Sigma \Rightarrow \varphi \approx \psi$ a $\mathcal{L}$-quasiequation.

For a class $\mathcal{K}$ of $\mathcal{L}$-algebras, an $\mathcal{L}$-quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is

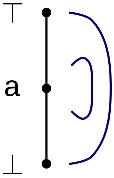
$\mathcal{K}$-valid, $\Sigma \models_{\mathcal{K}} \varphi \approx \psi$, if for every algebra $\mathbf{A} \in \mathcal{K}$ and every homomorphism $h\colon \mathbf{Tm}_{\mathcal{L}} \to \mathbf{A}$:
$h(\varphi') = h(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ implies $h(\varphi) = h(\psi)$.

$\mathcal{K}$-admissible if for every homomorphism $\sigma\colon \mathbf{Tm}_{\mathcal{L}} \to \mathbf{Tm}_{\mathcal{L}}$:
$\models_{\mathcal{K}} \sigma(\varphi') \approx \sigma(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ implies $\models_{\mathcal{K}} \sigma(\varphi) \approx \sigma(\psi)$.

If $\mathcal{K} = \{\mathbf{A}\}$, we usually write **A**-valid and **A**-admissible.

Example: Kleene Lattice KL

Consider the Kleene lattice **KL** $= \langle \{\bot, \mathrm{a}, \top\}, \wedge, \vee, \neg \rangle$:



Then (since no term is constantly a)

$$\{x \approx \neg x\} \;\Rightarrow\; x \approx y$$

is **KL**-admissible, but not **KL**-valid. The same holds for the following quasiequation ($x \preceq y$ stands for $x \approx x \wedge y$):

$$\{\neg x \preceq x,\, x \wedge \neg y \preceq \neg x \vee y\} \;\Rightarrow\; \neg y \preceq y$$

Free Algebras

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. Then the quotient algebra

$$\mathbf{F}_{\mathcal{K}}(X) \;=\; \mathbf{Tm}_{\mathcal{L}}(X) \,/ \sim_{\mathcal{K}} \qquad (\text{with } \varphi \sim_{\mathcal{K}} \psi \text{ iff } \models_{\mathcal{K}} \varphi \approx \psi)$$

with universe $\mathrm{F}_{\mathcal{K}}(X) = \{[\varphi]_{\sim_{\mathcal{K}}} \mid \varphi \in \mathrm{Tm}_{\mathcal{L}}(X)\}$ and operations

$$f([\varphi_1]_{\sim_{\mathcal{K}}}, \ldots, [\varphi_n]_{\sim_{\mathcal{K}}}) = [f(\varphi_1, \ldots, \varphi_n)]_{\sim_{\mathcal{K}}}$$

is called the X-generated free algebra of $\mathcal{K}$.

In particular, $\mathbf{F}_{\mathcal{K}}(n)$ is the free algebra on n generators of $\mathcal{K}$, and if $m = \max\{|A| : \mathbf{A} \in \mathcal{K}\}$, then for all $\varphi, \psi \in \mathrm{Tm}_{\mathcal{L}}(m)$:

$$\models_{\mathcal{K}} \varphi \approx \psi \qquad \text{iff} \qquad \models_{\mathbf{F}_{\mathcal{K}}(m)} \varphi \approx \psi.$$

Free Algebras and Admissibility

Theorem (Rybakov)

Let $\mathcal{K}$ be a finite set of finite $\mathcal{L}$-algebras, $n = \max\{|A| : \mathbf{A} \in \mathcal{K}\}$, $\Sigma \Rightarrow \varphi \approx \psi$ a quasiequation. The following are equivalent:

1. $\Sigma \Rightarrow \varphi \approx \psi$ *is $\mathcal{K}$-admissible.*
2. $\Sigma \Rightarrow \varphi \approx \psi$ *is $\mathbb{Q}(\mathcal{K})$-admissible.*
3. $\Sigma \Rightarrow \varphi \approx \psi$ *is $\mathbf{F}_{\mathcal{K}}(n)$-valid.*

Moreover, $\mathbf{F}_{\mathcal{K}}(n)$ is finite, so checking $\mathcal{K}$-admissibility is decidable. But $\mathbf{F}_{\mathcal{K}}(n)$ usually is very big, e.g., $|\mathbf{F}_{\mathbf{C_3}}(3)| = 43916$.

We seek a set of algebras $\mathcal{K}'$, called admissibility set of $\mathcal{K}$, s.t.

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathcal{K}\text{-admissible} \quad \text{iff} \quad \Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathcal{K}'\text{-valid.}$$

Checking Admissibility

Theorem

Given a class of algebras $\mathcal{K}$, the following are equivalent:

1. $\mathcal{K}' \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{K}}(\omega))$ *and* $\mathcal{K} \subseteq \mathbb{V}(\mathcal{K}')$.
2. $\mathbb{Q}(\mathcal{K}') = \mathbb{Q}(\mathbf{F}_{\mathcal{K}}(\omega))$.

Corollary

Given a finite set $\mathcal{K}$ of finite algebras, every set $\mathcal{K}'$ with $\mathcal{K}' \subseteq \mathbb{S}(\mathbf{F}_{\mathcal{K}}(\omega))$ and $\mathcal{K} \subseteq \mathbb{H}(\mathcal{K}')$ is an admissibility set of $\mathcal{K}$, i.e.,

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathcal{K}\text{-admissible} \quad \text{iff} \quad \Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathcal{K}'\text{-valid.}$$

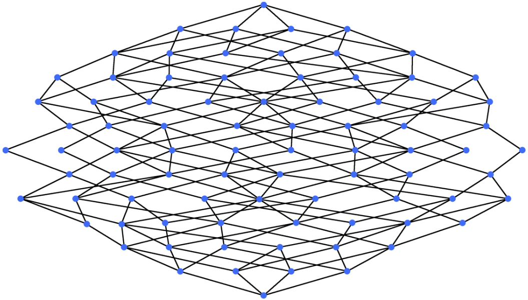
A Possible Algorithm

function ADMALGS($\mathcal{K}$)
    declare $\mathcal{A}, \mathcal{D}$: set
    declare $\mathbf{B}, \mathbf{B}'$: algebra
    $\mathcal{A} \leftarrow \emptyset$
    for all $\mathbf{A} \in \mathcal{D}$ **do**
        $\mathbf{B} \leftarrow$ FREE($\mathbf{A}, \mathcal{D}$)
        $\mathbf{B}' \leftarrow$ SUBPREHOM($\mathbf{A}, \mathbf{B}$)
        while $\mathbf{B}' \neq \mathbf{B}$ **do**
            $\mathbf{B} \leftarrow \mathbf{B}'$
            $\mathbf{B}' \leftarrow$ SUBPREHOM($\mathbf{A}, \mathbf{B}$)
        end while
        add $\mathbf{B}$ to $\mathcal{A}$
    end for
    return $\mathcal{A}$
end function

Example: Kleene lattice KL

Let us again look at the algebra **KL**.

1. **KL** $\notin \mathbb{H}(\mathbf{F_{KL}}(1))$, but **KL** $\in \mathbb{H}(\mathbf{F_{KL}}(2))$.
2. $\mathbf{F_{KL}}(2)$ has 82 elements and the smallest subalgebras $\mathbf{B} \leq \mathbf{F_{KL}}(2)$ with **KL** $\in \mathbb{H}(\mathbf{B})$ have 4 elements.



Minimal Generating Set

A set of finite algebras $\mathcal{K} = \{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ is called a minimal generating set for the quasivariety $\mathbb{Q}(\mathcal{K})$ if for every set $\mathcal{K}' = \{\mathbf{B}_1, \dots, \mathbf{B}_k\}$:

$$\mathbb{Q}(\mathcal{K}) = \mathbb{Q}(\mathcal{K}') \quad \text{implies} \quad [|A_1|, \dots, |A_n|] \leq_m [|B_1|, \dots, |B_k|].$$

Theorem

If $\mathcal{Q} = \mathbb{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n)$, $\mathbf{A}_1, \dots, \mathbf{A}_n$ are $\mathcal{Q}$-subdirectly irreducible finite algebras, and $\mathbf{A}_i \notin \mathbb{IS}(\mathbf{A}_j)$ for all $i \neq j$, then $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ is a minimal generating set for $\mathcal{Q}$. Moreover, this is the unique minimal generating set for $\mathcal{Q}$ up to isomorphism.

AdmAlgs

function AdmAlgs($\mathcal{K}$)
  declare $\mathcal{A}, \mathcal{D}$: set
  declare $\mathbf{B}, \mathbf{B}'$: algebra
  $\mathcal{D} \leftarrow$ MinGenSet($\mathcal{K}$)
  $\mathcal{A} \leftarrow \emptyset$
  for all $\mathbf{A} \in \mathcal{D}$ **do**
    $\mathbf{B} \leftarrow$ Free($\mathbf{A}, \mathcal{D}$)
    $\mathbf{B}' \leftarrow$ SubPreHom($\mathbf{A}, \mathbf{B}$)
    while $\mathbf{B}' \neq \mathbf{B}$ **do**
      $\mathbf{B} \leftarrow \mathbf{B}'$
      $\mathbf{B}' \leftarrow$ SubPreHom($\mathbf{A}, \mathbf{B}$)
    end while
    add $\mathbf{B}$ to $\mathcal{A}$
  end for
  return MinGenSet($\mathcal{A}$)
end function

Structural and Almost Structural Completeness

Theorem

The following are equivalent for any finite set $\mathcal{K}$ of finite $\mathcal{L}$-algebras and $n = \max\{|C| : \mathbf{C} \in \mathcal{K}\}$:

1. *$\mathcal{K}$ is structurally complete.*
2. $\textsc{MinGenSet}(\mathcal{K}) \subseteq \mathbb{IS}(\mathbf{F}_{\mathcal{K}}(n))$.

Theorem

The following are equivalent for any finite set $\mathcal{K}$ of finite $\mathcal{L}$-algebras, $\mathbf{B} \in \mathbb{S}(\mathbf{F}_{\mathcal{K}}(\omega))$ and $n := \max\{|C| : \mathbf{C} \in \mathcal{K}\}$:

1. *$\mathcal{K}$ is almost structurally complete.*
2. $\textsc{MinGenSet}(\{\mathbf{A} \times \mathbf{B} : \mathbf{A} \in \mathcal{K}\}) \subseteq \mathbb{IS}(\mathbf{F}_{\mathcal{K}}(n))$.

Some Experiments

A	$\lvert A \rvert$	Language	Quasivariety $\mathbb{Q}(\mathbf{A})$	n	$\mathbf{F}(n)$	M	SC	Reduction
BA	2	$\wedge, \vee, \neg, \bot, \top$	Boolean algebras	0	2	2	sc	0 %
$\mathbf{PCL}_2$	5	$\wedge, \vee, ^*, \bot, \top$	$\mathbb{Q}(\mathbf{PCL}_2)$	1	7	5	sc	29 %
$\mathbf{PCL}_1$	3	$\wedge, \vee, ^*, \bot, \top$	Stone algebras	1	6	3	sc	50 %
$\mathbf{L_3}$	3	$\to, \neg$	Algebras for Ł$_3$	1	12	6	asc	50 %
P	4	$*$	$\mathbb{Q}(\mathbf{P})$	2	6	3	sc	50 %
$\mathbf{G_{106}}$	3	$\circ$	$\mathbb{Q}(\mathbf{G_{106}})$	2	10	2,2	no	80 %
$\mathbf{M_5}$	5	$\wedge, \vee$	Lattices in $\mathbb{Q}(\mathbf{M_5})$	3	28	5	sc	82 %
$\mathbf{L_3}^{\to}$	3	$\to$	Algebras for Ł$_3^{\to}$	2	40	3	sc	93 %
$\mathbf{Z_3}^{\to}$	3	$\to$	Algebras for RM$^{\to}$	2	60	3	sc	95 %
KL	3	$\wedge, \vee, \neg$	Kleene lattices	2	82	4	no	95 %
$\mathbf{N_5}$	5	$\wedge, \vee$	Lattices in $\mathbb{Q}(\mathbf{N_5})$	3	99	5	sc	95 %
$\mathbf{PCL}_3$	9	$\wedge, \vee, ^*, \bot, \top$	$\mathbb{Q}(\mathbf{PCL}_3)$	2	625	19	no	97 %
$\mathbf{Z_3}^{\to\neg}$	3	$\to, \neg$	Algebras for RM$^{\to\neg}$	2	264	6	asc	98 %
$\mathbf{Z_3}$	3	$\wedge, \vee, \to, \neg$	$\mathbb{Q}(\mathbf{Z_3})$	2	1296	6	asc	100 %
$\mathbf{Lat}_8$	5	$\wedge, \vee$	$\mathbb{Q}(\mathbf{Lat}_8)$	5	7579	2	sc	100 %

Thank you for your attention!